Bachelorarbeit

Spinor formulation of Lanczos potentials in Riemannian Geometry

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Abstract

The conformal curvature C_{abcd} of a spacetime admits a potential L_{abc}, known as the Lanczos potential. The topic of this thesis is the formulation of the Lanczos potential in the spinor formalism, where the high symmetry of the Weyl spinor Ψ_{ABCD} simplifies many calculations. Various properties of the Lanczos spinor are discussed, the most important of which is the Illge wave equation, which can be leveraged for a proof of existence. The formulation in the Newman-Penrose and Geroch-Held-Penrose formalisms is also considered. Finally, spinorial solutions of the Lanczos potential for some classes of algebraically special spacetimes are discussed.

Zusammenfassung

Die konforme Krümming C_{abcd} einer Raumzeit besitzt ein Potential L_{abc}, welches als Lanczos-Potential bekannt ist. Thema dieser Arbeit ist die Formulierung des Lanczos-Potentials im Spinor-Formalismus, wo die hohe Symmetrie des Weyl-Spinors Ψ_{ABCD} viele Berechnungen vereinfacht. Verschiedene Eigenschaften des Lanczos-Spinors werden diskutiert, von denen die wichtigste die Illge-Wellengleichung ist, welche auch zum Beweis der Existenz des Potentials verwendet werden kann. Die Formulierung in den Newman-Penrose und Geroch-Held-Penrose Formalismen wird ebenfalls behandelt. Zuletzt werden spinorielle Lösungen für das Lanczos-Potential von einigen Klassen von algebraisch speziellen Raumzeiten diskutiert.

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Chapter 1: Introduction

Under Einstein's theory of general relativity, spacetime is represented through a pseudo-Riemannian manifold (M, g) of Lorentzian signature. The curvature of such a spacetime, as represented by the Riemann curvature tensor R_{abcd}, may be decomposed into two components: the Ricci curvature $R_{ab} := R^c{}_{acb}$ and the trace-free Weyl conformal curvature C_{abcd}. While the Ricci curvature is determined, through the Einstein field equations

$$R_{ab} - \tfrac{1}{2} R g_{ab} = \kappa T_{ab}, \quad R := R^a{}_a, \quad \kappa := \frac{8\pi G}{c^4}$$

by the energy-momentum tensor T_{ab}, it is less clear precisely how the Weyl curvature is generated. Based on the circumstance that vacuum spacetimes ($T_{ab} = 0$) are Ricci-flat and the conformal curvature is thus their only non-vanishing component, we may deduce that it is determined by the symmetries, as well as boundary and continuity conditions imposed on the spacetime.

In 1962 Lanczos [21] found that the Weyl tensor can be explicitly generated in a differential way from a third order potential L_{abc} with symmetries $L_{abc} = L_{[ab]c}$, $L_{[abc]} = 0$, by

$$\begin{aligned} C_{abcd} = {} & 2L_{ab[c;d]} + 2L_{cd[a;b]} \\ & - (g_{ac} L_{(bd)} - g_{ad} L_{(bc)} + g_{bd} L_{(ac)} - g_{bc} L_{(ad)}) \\ & + \tfrac{4}{3} L_{kl}{}^{k;l} g_{a[c} g_{d]b}, \end{aligned} \tag{1.1}$$

where $L_a{}^b := 2L_{ak}{}^{[b;k]}$. Equation (1.1) is known as the Weyl-Lanczos equation, while L_{abc} is the Lanczos potential or spintensor. It is notable that the existence of such a potential does not depend on the role of C_{abcd} as the conformal curvature, but is a result of its algebraic symmetries. In fact, a potential of this kind exists for any tensor W_{abcd} satisfying

$$W_{abcd} = -W_{bacd} = -W_{abdc} = W_{cdab}, \;\; W_{a[bcd]} = 0 \;\text{ and }\; W^c{}_{acb} = 0.$$

While the symmetries of the Weyl curvature are rather complicated in the tensor formalism, the corresponding spinorial object Ψ_{ABCD} is fully symmetric with $\Psi_{ABCD} = \Psi_{(ABCD)}$, which suggests that a spinor formulation of the Lanczos potential may be significantly simpler. Indeed, the spinorial Weyl-Lanczos equation reduces to

$$\Psi_{ABCD} = 2\nabla^{A'}{}_{(A} L_{BCD)A'},$$

where $L_{ABCA'} = L_{(ABC)A'}$ is the Lanczos spinor. This spinorial formulation, as well as related properties and solutions, are the topic of this thesis.

This work is structured as follows: First, the remainder of this chapter contains a brief overview of the spinor formalism, summarizing relevant identities and notational conventions. Chapter 2 begins with a more detailed description of the Lanczos potential in the tensor formalism, followed by a conversion into the spinor formalism and a discussion of various general properties that may be derived with it. Additionally, the problem is also formulated in the Newman-Penrose (NP) and Geroch-Held-Penrose (GHP) formalisms. Chapter 3 discusses spinorial solutions of the Weyl-Lanczos equation for certain spacetime classes. Finally, chapter 4 summarizes and briefly discusses physical significance.

1.1 Spinor formalism

As we will be using the two-component spinor formalism extensively, this section provides a brief summary of important identities and notational conventions. Spinors may be introduced in a number of ways, here we are mostly considering them as an algebraic construction, and will forgo a discussion of geometric interpretation and group-theoretic underpinnings. Unless otherwise mentioned, we are following the conventions of Penrose and Rindler [25]. The spacetime signature is $(+,-,-,-)$.

A spin-space is a two-dimensional module $\mathfrak{S}^A$ over C^∞ complex fields, together with an antisymmetric 2-form $\epsilon_{AB} = -\epsilon_{BA}$ acting as the metric. The metric defines a canonical isomorphism with the dual space $\mathfrak{S}_A$ by

$$\xi_B = \epsilon_{AB}\xi^A, \quad \xi^A = \epsilon^{AB}\xi_B,$$

where, due to the anti-symmetry of the ϵ-spinor, some care has to be taken regarding the index ordering conventions. Additionally, complex conjugation defines a canonical anti-isomorphism with the primed spaces $\mathfrak{S}^{A'}$ and $\mathfrak{S}_{A'}$ by

$$\overline{\xi^A} = \bar{\xi}^{A'}, \quad \overline{\xi_A} = \bar{\xi}_{A'}.$$

The spinor indexes $A, B, \dots$ and $A', B', \dots$ are abstract indexes, in that they do not refer to any particular basis. Spinor components in a specific basis will be written fat, such as $\xi^{\mathbf{A}}$. A tensor index a is identified with the spinor index pair AA'. Because these are abstract indexes, the Infeld-van der Waerden symbols $g_{\mathbf{a}}{}^{\mathbf{AA'}}$ do not appear.

Due to the low dimensionality of spin-spaces, anti-symmetrization over three or more indexes vanishes. In particular $\epsilon_{A[B}\epsilon_{CD]} = 0$, from which follows the useful identity

$$2\chi_{\cdots[AB]} = \chi_{\cdots C}{}^{C}\epsilon_{AB},$$

which also implies that any trace-free index pair is symmetric.

The ϵ-spinor relates to the spacetime metric by $g_{ab} = \epsilon_{AB}\epsilon_{A'B'}$. The Levi-Civita tensor is given by

$$e_{abcd} = i\epsilon_{AC}\epsilon_{BD}\epsilon_{A'D'}\epsilon_{B'C'} - i\epsilon_{AD}\epsilon_{BC}\epsilon_{A'C'}\epsilon_{B'D'}$$

and the Hodge dual is defined as ${}^*F_{ab} = \frac{1}{2}e_{ab}{}^{cd}F_{cd}$. Under these conventions ${}^{**}F_{ab} = -F_{ab}$.

Moving on to differential properties, the Levi-Civita covariant derivative ∇_a can be extended in a unique way to act on spinors, such that it is compatible with ϵ_{AB}, that is $\nabla_{AA'}\epsilon_{BC} = 0$.

Based on its algebraic symmetries, the curvature tensor R_{abcd} may be decomposed into spinorial quantities as follows:

$$R_{abcd} = \underbrace{[\Psi_{ABCD} + \Lambda(\epsilon_{AD}\epsilon_{BC} + \epsilon_{AC}\epsilon_{BD})]}_{=:X_{ABCD}}\epsilon_{A'B'}\epsilon_{C'D'} + \Phi_{ABC'D'}\epsilon_{A'B'}\epsilon_{CD} + c.c. \tag{1.2}$$

Here Λ is real and relates to the scalar curvature by $R = 24\Lambda$, $\Phi_{ABA'B'} = \Phi_{(AB)(A'B')}$ is real and corresponds to the trace-free Ricci curvature, and $\Psi_{ABCD} = \Psi_{(ABCD)}$ is complex and corresponds to the conformal curvature. To be more specific, the Weyl tensor is given by

$$C_{abcd} = \Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \overline{\Psi}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD},$$

where the two terms are the anti-self-dual component Ψ and the self-dual component $\overline{\Psi}$ of the Weyl tensor, respectively.

By the Ricci identity, the Riemann tensor appears when commutators of ∇_a are applied to tensors. Similarly, the spinorial curvature components appear when applying certain commutators of the covariant spinor derivative. Defining the operators

$$\square_{AB} := \nabla_{X'(A}\nabla_{B)}{}^{X'}, \quad \square_{A'B'} := \nabla_{X(A'}\nabla_{B')}{}^{X}$$

we have that $\Box_{AB}$ acts on a spinor ϕ by

$$\Box_{AB}\phi^{C}{}_{D}{}^{E'}{}_{F'} = X_{ABQ}{}^{C}\phi^{Q}{}_{D}{}^{E'}{}_{F'} - X_{ABD}{}^{Q}\phi^{C}{}_{Q}{}^{E'}{}_{F'} + \Phi_{ABQ'}{}^{E'}\phi^{C}{}_{D}{}^{Q'}{}_{F'} - \Phi_{ABF'}{}^{Q'}\phi^{C}{}_{D}{}^{E'}{}_{Q'}$$

and the action of $\Box_{A'B'}$ may be obtained through complex conjugation. Additionally taking into account the definition of X_{ABCD} from equation (1.2), we have the two useful special cases

$$\Box_{(AB}\kappa_{C)} = -\Psi_{ABC}{}^{D}\kappa_{D} \quad \text{and} \quad \Box_{AB}\kappa^{B} = -3\Lambda\kappa_{A}.$$

The Bianchi identity $\nabla_{[a}R_{bc]de} = 0$ in the spinor formalism may be split into two equations

$$\nabla_{B'}{}^{A}\Psi_{ABCD} = \nabla^{A'}{}_{(B}\Phi_{CD)A'B'} \quad \text{and} \quad \nabla^{CA'}\Phi_{CDA'B'} + 3\nabla_{DB'}\Lambda = 0,$$

where for our purposes mostly the first is of interest. Notably, in vacuum spacetimes (and more generally, Einstein spacetimes) we have that $\nabla_{B'}{}^{A}\Psi_{ABCD} = 0$.

Finally, the Weyl spinor (like any totally symmetric spinor) may be decomposed as

$$\Psi_{ABCD} = \kappa^{1}_{(A}\kappa^{2}_{B}\kappa^{3}_{C}\kappa^{4}_{D)},$$

where the κ^{i}_{A} are referred to as principal spinors (with corresponding principal null directions). By the Petrov classification, we then say that Ψ is of a certain Petrov type, if no (type I), two (type II), two pairs (type D), three (type III) or four (type N) principal null directions coincide. In case of $\Psi = 0$ the spacetime is conformally flat (type O).

Chapter 2: Properties

This chapter will first provide an overview of Lanczos potentials in the tensor formalism, including some historical context. Next, the translation into the spinor formalism will be covered, and properties of the Lanczos spinor will be discussed. These include the Illge wave equation, which is also used to prove the existence and uniqueness of the potential. Finally, the formulation in the NP and GHP formalisms is considered.

2.1 Lanczos potentials in the tensor formalism

The object nowadays referred to as the Lanczos potential originates from a 1962 work by Lanczos [21], wherein the splitting of the Riemann tensor into its self-double-dual S and anti-self-double-dual A components

$$S_{abcd} = R_{abcd} + {}^*R^*{}_{abcd}, \quad A_{abcd} = R_{abcd} - {}^*R^*{}_{abcd}$$

is investigated. Lanczos observed that while the self-double-dual component[1] is generated by the Ricci tensor and the metric through

$$S_{abcd} = (R_{ac} - \tfrac{1}{4}Rg_{ac})g_{bd} + (R_{bd} - \tfrac{1}{4}Rg_{bd})g_{ac} - (R_{ad} - \tfrac{1}{4}Rg_{ad})g_{bc} - (R_{bc} - \tfrac{1}{4}Rg_{bc})g_{ad},$$

no such relationship for the anti-self-double-dual component is known. Lanczos endeavored to find a similar generating function for A_{abcd}. We will only sketch the derivation here, as the variational angle from which the problem is approached is not very relevant for our further considerations.

Lanczos considers a variational problem where ${}^*R^{*abcd}$ and g_{ab} are a priori independent variational variables. To obtain a first-order Lagrangian, Γ^c_{ab} is also included as a variational variable. Of course, these quantities are not actually independent, so that a number of constraints have to be enforced using Lagrange multipliers: these are the Bianchi identity ${}^*R^{*abcd}{}_{;d} = 0$, the use of the Levi-Civita connection $\Gamma^c_{ab} = \left\{{c \atop ab}\right\}$, as well as the particular relationship between the Christoffel symbols and the Ricci tensor. This yields the Lagrangian

$$\begin{aligned}\mathcal{L}' = \mathcal{L}({}^*R^{*abcd}, g_{ab}) + L_{abc}{}^*R^{*abcd}{}_{;d} + P^{ab}{}_c(\Gamma^c_{ab} - \left\{{c \atop ab}\right\}) \\ + \rho^{ab}(R_{ab} + \Gamma^c_{bc,a} - \Gamma^c_{ab,c} + \Gamma^c_{ad}\Gamma^d_{bc} - \Gamma^c_{ab}\Gamma^d_{dc}).\end{aligned}$$

Additionally, ${}^*R^{*abcd}$ has to satisfy the usual algebraic symmetries, which are not included in the Lagrangian, and will be imposed after the variation instead. L_{abc}, $P^{ab}{}_c$ and ρ^{ab} here are the Lagrange multipliers for the constraints on ${}^*R^{*abcd}$, Γ^c_{ab} and g_{ab}, where L_{abc} is anti-symmetric in ab, while $P^{ab}{}_c$ and ρ^{ab} are symmetric. ${}^*R^{*abcd}$ has 20 independent components, while its conjugate L_{abc} has 24. To match up the degrees of freedom, Lanczos imposed the (somewhat arbitrary) condition $L_{[abc]} = 0$.

The base Langrangian was chosen as $\mathcal{L} = \frac{1}{8}{}^*R^{*abcd}R_{abcd}$, which Lanczos has previously shown to vanish identically under variation of g_{ab} [20], thus avoiding the imposition of any

[1] Due to different sign conventions, the meaning of the self- and anti-self-double-dual parts is interchanged compared to Lanczos work.

particular field equations. Variation of $\mathcal{L}'$ with respect to ${}^*R^{*abcd}$, an application of its algebraic symmetries and a longer calculation yield an expression for R_{abcd} in terms of L_{abc}, R_{ab} and g_{ab} only.

However, the terms containing R_{ab} coincide with the trace terms that are excluded from the Weyl tensor. As such, the resulting expression for the Weyl tensor depends on L_{abc} (and the spacetime metric) only:

$$\begin{aligned} C_{abcd} = {} & 2L_{ab[c;d]} + 2L_{cd[a;b]} \\ & - (g_{ac}L_{(bd)} - g_{ad}L_{(bc)} + g_{bd}L_{(ac)} - g_{bc}L_{(ad)}) \\ & + \tfrac{4}{3}L_{kl}{}^{k;l} g_{a[c}g_{d]b}, \end{aligned} \tag{2.1}$$

with $L_a{}^b := 2L_{ak}{}^{[b;k]}$. This is precisely the Weyl-Lanczos equation already mentioned in the introduction, with L_{abc} referred to as the Lanczos potential and having symmetries $L_{abc} = L_{[ab]c}$ and $L_{[abc]} = 0$.

To return to the original motivation, the Weyl tensor is related to the anti-self-double-dual component of the Riemann tensor by $C_{abcd} = A_{abcd} - \frac{1}{6}R(g_{ac}g_{bd} - g_{ad}g_{bc})$. What has been achieved here, is a splitting of the Riemann tensor into two components, one that is generated in an algebraic manner from the Ricci tensor R_{ab}, and one that is generated in a differential manner from the Lanczos potential L_{abc}.

The Lanczos potential in this definition has 20 independent components, while the Weyl tensor only has 10. The 10 additional degrees of freedom are typically split into two gauge fields, the algebraic gauge χ^a and the differential gauge F^{ab}:

$$\chi^a = L^{ab}{}_b \tag{2.2}$$

$$F^{ab} = L^{abc}{}_{;c}. \tag{2.3}$$

The particular choices of $\chi^a = 0$ and $F^{ab} = 0$ are known as the Lanczos algebraic gauge and Lanczos differential gauge respectively. Publications will commonly assume that Lanczos potentials satisfy one or both of these, sometimes without specifying this explicitly. While the Lanczos algebraic gauge can always be easily achieved through the transformation

$$\hat{L}_{abc} = L_{abc} - \tfrac{1}{3}\chi_a g_{bc} + \tfrac{1}{3}\chi_b g_{ac},$$

which yields $\hat{L}_{ab}{}^b = 0$ while leaving the Weyl-Lanczos equation invariant, there is no general way of transforming a Lanczos potential to satisfy the Lanczos differential gauge.

In 1982 Bampi and Caviglia [7] pointed out that Lanczos' derivation does not constitute a proof of existence: Lanczos has shown that equation (2.1) is the Euler-Lagrange equation of a certain functional, but this does not by itself guarantee consistency. Bampi and Caviglia proceeded to provide a local existence proof using an entirely different method, namely Cartan's local criteria of integrability of ideals of exterior forms.

Additionally, they showed a number of important facts: First, the existence of the Lanczos potential does not depend on the geometric meaning of the Weyl tensor. Instead, any tensor W_{abcd} satisfying the algebraic symmetries of the Weyl tensor

$$W_{abcd} = -W_{bacd} = -W_{abdc} = W_{cdab}, \;\; W_{a[bcd]} = 0 \text{ and } W^c{}_{acb} = 0, \tag{2.4}$$

admits a Lanczos potential. To clearly distinguish this case, W_{abcd} is commonly referred to as a "Weyl tensor candidate" and L_{abc} as a "Lanczos potential candidate".

Second, the differential gauge $F^{ab} = F^{[ab]}$ can indeed be fixed arbitrarily, without affecting existence. This justifies the validity of imposing the Lanczos differential gauge $L^{abc}{}_{;c} = 0$, though it should be noted that imposing this gauge condition may make it significantly harder to find *explicit* solutions. And finally, they showed that a similar (analytic) potential for the full Riemann tensor (that is, if the last condition in (2.4) is relaxed) does not exist in the general case.

The Weyl-Lanczos equation (2.1) may also be written in the alternative form [27]

$$C_{abcd} = L_{ab[c;d]} + L_{cd[a;b]} - {}^*L^*_{ab[c;d]} - {}^*L^*_{cd[a;b]} - \tfrac{2}{3} L_{kl}{}^{k;l} g_{a[c} g_{d]b}. \tag{2.5}$$

While this form is less convenient for actual calculations, it highlights the structure of the construction, and makes certain symmetries more evident. For example, it is clearly visible that C_{abcd} is anti-self-double-dual, and that $C_{abcd} = -C_{bacd} = -C_{abdc} = C_{cdab}$ holds.

2.2 Lanczos potentials in the spinor formalism

We will now translate the tensorial objects and equations from the previous section into their spinorial counterparts. A spinorial formulation of the Lanczos potential was first given by Maher and Zund in 1968 [23]. Some authors attribute the first spinor formulation to Taub instead, who (independently) provided a spinor formulation in 1975 [29]. We follow the general approach of Maher and Zund here, though we will refrain from imposing the Lanczos gauge conditions initially.

Lanczos spinor Due to the skew symmetry $L_{abc} = L_{[ab]c}$ and reality of the Lanczos potential, we may decompose it into[2]

$$L_{abc} = L_{AA'BB'CC'} = L_{ABCC'}\epsilon_{A'B'} + \overline{L}_{A'B'C'C}\epsilon_{AB},$$

where

$$L_{ABCC'} := \tfrac{1}{2} L_{AA'B}{}^{A'}{}_{CC'} \quad \text{satisfies} \quad L_{ABCC'} = L_{(AB)CC'}.$$

The cyclic property $L_{[abc]} = 0$ may be more conveniently written as ${}^*L_{ab}{}^b = {}^*L_{abc}g^{bc} = 0$, which implies

$$\begin{aligned} 0 &= (-iL_{ABCC'}\epsilon_{A'B'} + i\overline{L}_{A'B'C'C}\epsilon_{AB})\epsilon^{BC}\epsilon^{B'C'} \\ &= iL_{ABCA'}\epsilon^{BC} - i\overline{L}_{A'B'C'A}\epsilon^{B'C'} = iL_{AB}{}^B{}_{A'} - i\overline{L}_{A'B'}{}^{B'}{}_A. \end{aligned}$$

Defining $L_{AA'} := L_{AB}{}^B{}_{A'}$ this reduces to the constraint $L_{AA'} = \overline{L}_{AA'}$. The Lanczos algebraic gauge $L_{ab}{}^b = 0$ corresponds to

$$0 = L_{AA'BB'}{}^{BB'} = L_{AB}{}^{BB'}\epsilon_{A'B'} + \overline{L}_{A'B'}{}^{B'B}\epsilon_{AB} = -L_{AB}{}^B{}_{A'} - \overline{L}_{A'B'}{}^{B'}{}_A,$$

that is $L_{AA'} = -\overline{L}_{AA'}$. Together with the circular property this is only satisfied by $L_{AA'} = L_{AB}{}^B{}_{A'} = 0$, which implies that $L_{ABCC'} = L_{A(BC)C'}$. With the existing symmetry on the first index pair, this makes L fully symmetric with $L_{ABCC'} = L_{(ABC)C'}$.

The increased symmetry simplifies many calculations and an arbitrary potential can always be easily converted into the Lanczos algebraic gauge. Thus the general convention in the literature is to include this gauge condition in the definition of the Lanczos spinor. Unless explicitly noted otherwise, we will follow this convention.

[2] To stay consistent with modern conventions, the definition of the Lanczos spinor differs from Maher and Zund by a factor of two.

Weyl-Lanczos equation For the conversion of the Weyl-Lanczos equation we will briefly return to the more general form of the Lanczos spinor that does not include the Lanczos algebraic gauge. Writing out equation (2.1) in spinor components yields

$$
\begin{aligned}
C_{abcd} &= \Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \overline{\Psi}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD} \\
&= \nabla_{DD'}(L_{ABCC'}\epsilon_{A'B'} + \overline{L}_{A'B'C'C}\epsilon_{AB}) - \nabla_{CC'}(L_{ABDD'}\epsilon_{A'B'} + \overline{L}_{A'B'D'D}\epsilon_{AB}) \\
&\quad + \nabla_{BB'}(L_{CDAA'}\epsilon_{C'D'} + \overline{L}_{C'D'A'A}\epsilon_{CD}) - \nabla_{AA'}(L_{CDBB'}\epsilon_{C'D'} + \overline{L}_{C'D'B'B}\epsilon_{CD}) \\
&\quad - \tfrac{1}{2}\epsilon_{AC}\epsilon_{A'C'}(L_{BB'DD'} + L_{DD'BB'}) + \tfrac{1}{2}\epsilon_{AD}\epsilon_{A'D'}(L_{BB'CC'} + L_{CC'BB'}) \\
&\quad - \tfrac{1}{2}\epsilon_{BD}\epsilon_{B'D'}(L_{AA'CC'} + L_{CC'AA'}) + \tfrac{1}{2}\epsilon_{BC}\epsilon_{B'C'}(L_{AA'DD'} + L_{DD'AA'}) \\
&\quad + \tfrac{2}{3}\nabla_{LL'}L^{KK'LL'}{}_{KK'}(\epsilon_{AC}\epsilon_{DB}\epsilon_{A'C'}\epsilon_{D'B'} - \epsilon_{AD}\epsilon_{CB}\epsilon_{A'D'}\epsilon_{C'B'}),
\end{aligned}
$$

where it should be noted that $L_{ABCC'}$ refers to the Lanczos spinor, while $L_{AA'BB'}$ refers to the shorthand L_{ab}. Transvecting with $\epsilon^{A'B'}\epsilon^{C'D'}$ results in

$$
\begin{aligned}
4\Psi_{ABCD} &= 2\nabla_{D}{}^{C'}L_{ABCC'} + 2\nabla_{C}{}^{D'}L_{ABDD'} + 2\nabla_{B}{}^{A'}L_{CDAA'} + 2\nabla_{A}{}^{B'}L_{CDBB'} \\
&\quad - \tfrac{1}{2}\epsilon_{AC}\epsilon^{B'D'}(L_{BB'DD'} + L_{DD'BB'}) - \tfrac{1}{2}\epsilon_{AD}\epsilon^{B'C'}(L_{BB'CC'} + L_{CC'BB'}) \\
&\quad - \tfrac{1}{2}\epsilon_{BD}\epsilon^{A'C'}(L_{AA'CC'} + L_{CC'AA'}) - \tfrac{1}{2}\epsilon_{BC}\epsilon^{A'D'}(L_{AA'DD'} + L_{DD'AA'}) \\
&\quad - \tfrac{4}{3}\nabla_{LL'}L^{KK'LL'}{}_{KK'}(\epsilon_{AC}\epsilon_{DB} + \epsilon_{AD}\epsilon_{CB}).
\end{aligned} \tag{2.6}
$$

Without using the algebraic gauge condition, simplifying this expression explicitly is rather tedious. As such, a derivation using explicit algebraic manipulation has been relayed to Appendix A.1. We can however arrive at the same result using a simple symmetry argument: Using $\Psi_{ABCD} = \Psi_{(ABCD)}$ we may symmetrize the right-hand side of the equation, in which case all terms involving the anti-symmetric ϵ_{AB} spinor vanish, while the first four terms become identical. As such, this expression reduces to

$$\Psi_{ABCD} = 2\nabla_{(A}{}^{A'}L_{BCD)A'}, \tag{2.7}$$

the spinorial Weyl-Lanczos equation.

Gauge conditions Clearly the Weyl-Lanczos equation stays invariant under gauge transformations of the type

$$\hat{L}_{ABCA'} = L_{ABCA'} + \epsilon_{AC}\chi_{BA'} + \epsilon_{BC}\chi_{AA'}, \tag{2.8}$$

where $\chi_{AA'}$ is real (to preserve the circular property). Noting that

$$\hat{L}_{AB}{}^{B}{}_{A'} = L_{AB}{}^{B}{}_{A'} + 3\chi_{AA'},$$

a choice of $\chi_{AA'} = -\frac{1}{3}L_{AB}{}^{B}{}_{A'}$ will result in a Lanczos spinor in Lanczos algebraic gauge. This reaffirms the earlier choice to include the algebraic gauge condition in the definition of the Lanczos spinor, and we will return to this convention now.

The more interesting gauge freedom is given by the differential gauge $F^{ab} = L^{abc}{}_{;c}$. Using anti-symmetry and reality we may write

$$F^{AA'BB'} = F^{AB}\epsilon^{A'B'} + \overline{F}^{A'B'}\epsilon^{AB},$$

with $F_{AB} = F_{(AB)}$, so that the gauge condition becomes

$$F^{AB}\epsilon^{A'B'} + \overline{F}^{A'B'}\epsilon^{AB} = \nabla_{CC'}(L^{ABCC'}\epsilon^{A'B'} + \overline{L}^{A'B'C'C}\epsilon^{AB}).$$

Transvecting with $\epsilon_{A'B'}$ yields

$$F^{AB} = \nabla_{CC'}L^{ABCC'}.$$

We observe that

$$\epsilon_{AB}F_{CD} = \epsilon_{AB}\nabla^{EE'}L_{CDEE'} = \nabla_B{}^{E'}L_{CDAE'} - \nabla_A{}^{E'}L_{CDBE'},$$

and consequently

$$\epsilon_{A(B}F_{CD)} = \nabla_{(B}{}^{E'}L_{CD)AE'} - \nabla_A{}^{E'}L_{BCDE'}.$$

As such, the Weyl-Lanczos equation may be written as

$$\begin{aligned}\Psi_{ABCD} &= 2\nabla_{(A}{}^{E'}L_{BCD)E'} = \tfrac{1}{2}\nabla_A{}^{E'}L_{BCDE'} + \tfrac{3}{2}\nabla_{(B}{}^{E'}L_{CD)AE'} \\ &= 2\nabla_A{}^{E'}L_{BCDE'} + \tfrac{3}{2}\epsilon_{A(B}F_{CD)}. \end{aligned} \tag{2.9}$$

This form will be simpler for some of the following calculations. It is also noteworthy that for the Lanczos differential gauge $F_{AB} = 0$, the Weyl-Lanczos equation reduces to

$$\Psi_{ABCD} = 2\nabla_A{}^{A'}L_{BCDA'}.$$

2.3 Wave equation, existence and uniqueness

One of the most significant results that have been first derived using the spinor formalism is the Illge wave equation [19], which in vacuum and Lanczos differential gauge may be written $\Box L_{ABCC'} = 0$, and thus also $\Box L_{abc} = 0$. A more complicated wave equation derived using the tensor formalism

$$\Box L_{abc} = 2L^{def}g_{[a|c|}C_{b]def} - 2L_{[a}{}^{de}C_{b]edc} - \tfrac{1}{2}L^{de}{}_cC_{deab}$$

was previously known, but based on Illge's result it was then found that the right hand side of this equation vanishes identically in four dimensions [13]. In this section, we will discuss the wave equation, an existence and uniqueness proof based on it, as well as some conceptually related superpotential constructions.

2.3.1 Wave equation

Differentiating the Weyl-Lanczos equation as written in equation (2.9) yields

$$\begin{aligned}\nabla^{AA'}\Psi_{ABCD} &= 2\nabla^{AA'}\nabla_A{}^{E'}L_{BCDE'} + \tfrac{3}{2}\nabla^{AA'}\epsilon_{A(B}F_{CD)} \\ &= 2\nabla^{A[A'}\nabla_A{}^{E']}L_{BCDE'} + 2\nabla^{A(A'}\nabla_A{}^{E')}L_{BCDE'} + \tfrac{3}{2}\nabla_{(B}{}^{A'}F_{CD)} \\ &= -\epsilon^{A'E'}\Box L_{BCDE'} - 2(\Phi^{A'E'E}{}_BL_{ECDE'} + \Phi^{A'E'E}{}_CL_{BEDE'} \\ &\qquad + \Phi^{A'E'E}{}_DL_{BCEE'} + 3\Lambda\epsilon^{A'E'}L_{BCDE'}) + \tfrac{3}{2}\nabla_{(B}{}^{A'}F_{CD)} \\ &= -\Box L_{BCD}{}^{A'} - 6\Phi^{A'E'E}{}_{(B}L_{CD)EE'} - 6\Lambda L_{BCD}{}^{A'} + \tfrac{3}{2}\nabla_{(B}{}^{A'}F_{CD)}. \end{aligned} \tag{2.10}$$

Per the spinorial Bianchi-identities it holds that $\nabla^A{}_{A'}\Psi_{ABCD} = \nabla_{(B}{}^{E'}\Phi_{CD)E'A'}$. Substituting this expression results in

$$\Box L_{BCDA'} + \nabla_{(B}{}^{E'}\Phi_{CD)E'A'} + 6\Phi_{A'E'E(B}L_{CD)}{}^{EE'} + 6\Lambda L_{BCDA'} - \tfrac{3}{2}\nabla_{A'(B}F_{CD)} = 0,$$

which is the general Illge wave equation. If the spacetime is vacuum ($\Phi_{ABA'B'} = 0$, $\Lambda = 0$) and the Lanczos spinor is in Lanczos differential gauge ($F_{AB} = 0$) this reduces to the previously mentioned form

$$\Box L_{ABCC'} = 0.$$

2.3.2 Existence and uniqueness

Existence and uniqueness (in a specific sense discussed in the following) of the Lanczos potential was first shown by Bampi and Caviglia [7]. However, their proof is mathematically rather involved and utilizes methods not commonly seen in general relativity: in broad terms, the Weyl-Lanczos equation is lifted into a system of exterior differential equations in 124 dimensions and then the existence of an involutive integral manifold is shown using local integrability criteria due to Cartan.

A simpler proof using the spinor formalism and the theory of hyperbolic systems was later provided by Illge [19]. Because of the reliance on the spinor formalism, this proof is valid only for spacetimes, while the proof of Bampi and Caviglia is signature-independent.

Theorem 1 (Existence and Uniqueness, Illge 1988)
Given spinor fields $W_{ABCD} = W_{(ABCD)}$ and $F_{AB} = F_{(AB)}$, as well as $\mathring{L}_{ABCC'} = \mathring{L}_{(ABC)C'}$ defined on a spacelike smooth hypersurface Σ only, there exists a neighborhood of Σ, in which the system

$$W_{ABCD} = 2\nabla_{(A}{}^{E'} L_{BCD)E'}$$
$$F_{BC} = \nabla^{AA'} L_{ABCA'}$$

has one and only one solution $L_{ABCC'} = L_{(ABC)C'}$ such that $L_{|\Sigma} = \mathring{L}$.

Of course, for $W_{ABCD} = \Psi_{ABCD}$ this corresponds to a solution of the Weyl-Lanczos equation. We will now outline Illge's proof, as it is rather straightforward and gives us some insight into the structure of the problem. First, we may rewrite this system in the form of equation (2.9):

$$W_{ABCD} = 2\nabla_A{}^{E'} L_{BCDE'} + \tfrac{3}{2}\epsilon_{A(B}F_{CD)}. \tag{2.11}$$

The proof will also make use of the wave equation (2.10), here reproduced for the general case using W instead of Ψ:

$$0 = \Box L_{BCDA'} + 6\Phi_{A'E'E(B}L_{CD)}{}^{EE'} + 6\Lambda L_{BCDA'} + \nabla_{A'}{}^{A} W_{ABCD} - \tfrac{3}{2}\nabla_{A'(B}F_{CD)}. \tag{2.12}$$

We now denote the future-directed unit normal field of Σ by n^a (with $n_a n^a = 1$) and the normal derivative operator by $\nabla_n := n^a \nabla_a$. The tangential derivative is then given by $\widetilde{\nabla}_a := \nabla_a - n_a \nabla_n$. Restricting equation (2.11) to Σ and expanding ∇_a yields

$$n_A{}^{E'} \nabla_n L_{BCDE'|\Sigma} = W_{ABCD|\Sigma} - 2\widetilde{\nabla}_A{}^{E'} L_{BCDE'|\Sigma} - \tfrac{3}{2}\epsilon_{A(B}F_{CD)|\Sigma}.$$

Multiplying by $n^A{}_{F'}$ and using that due to normalization $n_A{}^{E'} n^A{}_{F'} = \frac{1}{2}\epsilon^{E'}{}_{F'}$, we obtain an expression for $\nabla_n L_{|\Sigma}$ depending only on the values of W, F and L on the hypersurface. This means that given $\mathring{L}$, there is a unique $\nabla_n L_{|\Sigma}$, such that equation (2.11) is satisfied on Σ.

As we now have both $L_{|\Sigma} = \mathring{L}$ and $\nabla_n L_{|\Sigma}$ as initial data, the theory of hyperbolic systems guarantees that the wave equation (2.12), which is a linear, diagonal, second-order hyperbolic system, has a unique solution L in a neighborhood of Σ (see for example Theorem 10.1.2 in [34]). From the structure of the wave equation it is also clear that the solution must have symmetry $L_{ABCC'} = L_{(ABC)C'}$. What remains to be shown is that this L is also a solution of the Weyl-Lanczos equation (2.11) itself, or equivalently, that

$$\eta_{ABCD} := 2\nabla_A{}^{E'} L_{BCDE'} - W_{ABCD} + \tfrac{3}{2}\epsilon_{A(B}F_{CD)}$$

is everywhere vanishing. We already know that $\eta_{|\Sigma} = 0$, as equation (2.11) is satisfied on Σ. Additionally, L being a solution of equation (2.12) corresponds to $\nabla^{AA'}\eta_{ABCD} = 0$. Together with $\eta_{|\Sigma} = 0$ this implies that $\nabla_n L_{|\Sigma} = 0$ as well (from $\nabla_n = n^a(\nabla_a - \widetilde{\nabla}_a)$). Differentiating again we have:

$$\begin{aligned}0 &= \nabla_{A'E}\nabla^{A'}{}_A\eta^A{}_{BCD} = \nabla_{A'[E}\nabla^{A'}{}_{A]}\eta^A{}_{BCD} + \nabla_{A'(E}\nabla^{A'}{}_{A)}\eta^A{}_{BCD} \\ &= \tfrac{1}{2}\epsilon_{EA}\Box\eta^A{}_{BCD} - 3\Lambda\eta_{EBCD} - 3X_{EA(B}{}^F\eta^A{}_{CD)F} \\ &= -\tfrac{1}{2}\Box\eta_{EBCD} + 3\Psi_{AFE(B}\eta^{AF}{}_{CD)} - 3\Lambda[\eta_{EBCD} + \eta_{(BCD)E} + \epsilon_{E(B}\eta^F{}_{CD)F}].\end{aligned}$$

This is another another linear, diagonal, second-order hyperbolic system, as such the solution $\eta = 0$ satisfying $\eta_{|\Sigma} = \nabla_n\eta_{|\Sigma} = 0$ is indeed the only solution. As such, the obtained L also solves equation (2.11), concluding the proof.

The last step in the proof gives us some insight into the role of the wave equation: If we can satisfy the Weyl-Lanczos equation on a single hypersurface, then the wave equation is sufficient to extend this into a complete solution. We will further discuss the wave equation as a sufficient condition in section 2.3.4.

While the proven theorem only makes local statements (within a causal neighborhood to be precise), the result can be extended to a global statement for a relatively large class of spacetimes: Assuming existence of a global spin structure, if the spacetime is also globally hyperbolic and the hypersurface Σ is a Cauchy surface, then we have a globally unique solution (in the sense of Theorem 1).

Finally, it should be noted that the result proven by Illge is somewhat stronger than what was presented here. The proof generalizes in a straightforward manner to potentials of higher order, yielding the following theorem.

Theorem 2 (Generalized Existence and Uniqueness, Illge 1988)
Given a spacelike smooth hypersurface Σ and spinor fields

$$\begin{aligned}W_{AA_1\ldots A_nB'_1\ldots B'_m} &= W_{(AA_1\ldots A_n)(B'_1\ldots B'_m)} \\ F_{A_2\ldots A_nB'_1\ldots B'_m} &= F_{(A_2\ldots A_n)(B'_1\ldots B'_m)} \\ \mathring{L}_{A_1\ldots A_nB'_1\ldots B'_mA'} &= \mathring{L}_{(A_1\ldots A_n)(B'_1\ldots B'_m)A'} \quad \textit{on } \Sigma \textit{ only,}\end{aligned}$$

there exists a neighborhood of Σ, in which the system

$$\begin{aligned}W_{AA_1\ldots A_nB'_1\ldots B'_m} &= 2\nabla_{(A}{}^{A'}L_{A_1\ldots A_n)A'} \\ F_{A_2\ldots A_nB'_1\ldots B'_m} &= \nabla^{AA'}L_{AA_2\ldots A_nB'_1\ldots B'_mA'}\end{aligned}$$

has one and only one solution $L_{A_1\ldots A_nB'_1\ldots B'_mA'} = L_{(A_1\ldots A_n)(B'_1\ldots B'_m)A'}$ such that $L_{|\Sigma} = \mathring{L}$.

It should be emphasized, however, that in the presence of both primed and unprimed indexes, the resulting potential is no longer fully symmetric (in the above formulation, the symmetry does not include A'). Requiring complete symmetry imposes an additional algebraic constraint on the differential system.

2.3.3 Superpotentials

Given these fairly general results on the existence of spinor potentials, it should come as no surprise that it is possible to construct higher-order potentials for the Lanczos potential itself. Two such constructions have been investigated by Andersson and Edgar [4].

The first potential that can be constructed is

$$L_{ABCC'} = \nabla_{C'}{}^D T_{ABCD}, \tag{2.13}$$

with symmetry $T_{ABCD} = T_{(ABC)D}$. This potential always exists due to the complex conjugate of Theorem 2 (with $n = 0$ and $m = 3$). This potential does not appear to be particularly useful for finding Lanczos potentials, but it can be leveraged for a very simple existence proof, given in [4]: Supposing that L may be written in the form of equation (2.13) (but without making use of the known existence theorem) the Weyl-Lanczos equation becomes

$$W_{ABCD} = 2\nabla_{(A}{}^{E'} L_{BCD)E'} = 2\nabla_{(A|}{}^{E'} \nabla_{E'}{}^{E} T_{|BCD)E},$$

which may again be written as a linear, diagonal, second-order hyperbolic system

$$\begin{aligned}0 = \Box T_{BCDA} - 6\Psi_{A(B}{}^{EG} T_{CD)GE} + 6\Lambda(\epsilon_{A(B} T_{CD)E}{}^{E} + T_{A(BCD)} + T_{BCDA}) \\ - \tfrac{3}{2}\epsilon_{A(B} F_{CD)} + W_{ABCD},\end{aligned}$$

which exhibits the desired symmetries and must have a (local) solution. This proof is somewhat simpler than that of Illge, because the Weyl-Lanczos equation directly serves as the hyperbolic system, rather than using two derivatives of it as auxiliaries. On the other hand, this is only a proof of existence, it does not make any statement on uniqueness.

The second potential that may be constructed is

$$L_{ABCA'} = \nabla_{(A}{}^{B'} H_{BC)A'B'}, \tag{2.14}$$

with symmetry $H_{ABA'B'} = H_{(AB)(A'B')}$. Theorem 2 does not guarantee the existence of such a potential, because H is assumed to be fully symmetric here (including the primed indexes). In [4] it has been shown that such a potential always exists in Einstein spacetimes, where $\Phi_{ABA'B'} = 0$. However, certain non-Einstein spacetimes are known to have Lanczos potentials taking this form as well [2] [32]. As we will see in section 3.2, solutions of this form tend to have interesting properties.

2.3.4 Wave equation as a sufficient condition

For vacuum spacetimes and under Lanczos differential gauge, the wave equation (2.12) reduces to

$$0 = \Box L_{ABCA'} + \nabla_{A'}{}^{D} W_{ABCD}, \tag{2.15}$$

where $W_{ABCD} = 2\nabla_{(A}{}^{A'} L_{BCD)A'}$. If $W_{ABCD} = \Psi_{ABCD}$ then by the vacuum Bianchi identities we have that $\nabla_{A'}{}^{D} W_{ABCD} = 0$ and thus $\Box L_{ABCA'} = 0$. As such, $\Box L_{ABCA'} = 0$ is a *necessary* condition for L to be the Lanczos potential of the Weyl spinor Ψ in particular, rather than of some arbitrary "Weyl candidate" W. In [14] Edgar and Höglung now consider the question, under which circumstances it is also *sufficient*.

If $\Box L_{BCDA'} = 0$ then we have from equation (2.15) that $\nabla_{A'}{}^{A} W_{ABCD} = 0$. This type of equation has previously been analyzed by Bell and Szekeres [8], where it was found that for vacuum spacetimes of "sufficient generality" it is only solved by constant multiples of the Weyl spinor. Since a constant factor may be easily absorbed into L, we may say that in this case $\Box L_{ABCA'} = 0$ is indeed a sufficient condition to obtain a Lanczos potential of the Weyl spinor.

Vacuum spacetimes of "sufficient generality" here refer to algebraically general spacetimes that do not satisfy a certain strict condition given in [8]. If the condition is satisfied or if the spacetimes is algebraically special, then the solution of $\nabla_{A'}{}^{A} W_{ABCD} = 0$ may also have additional independent components. As the spacetimes we are usually interested in tend to be algebraically special, the wave equation $\Box L_{BCDA'} = 0$ is not particularly useful as a sufficient condition for practical purposes.

2.4 Behavior under conformal rescaling

As the potential of the conformal curvature, it is interesting to consider whether the Lanczos potential is well-behaved under conformal rescalings. A conformal rescaling

$$\hat{g}_{\mu\nu} = \Omega^2 g_{\mu\nu},$$

for a (nowhere vanishing) field Ω leaves the conformal curvature invariant, such that $\hat{\Psi}_{ABCD} = \Psi_{ABCD}$. The corresponding change of the ϵ-spinor is given by $\hat{\epsilon}_{AB} = \Omega\epsilon_{AB}$. To avoid ambiguity as to which ϵ-spinor is in use, all contractions will be made explicit in the following.

The covariant derivative transforms by [25, ch. 5.6]

$$\begin{aligned}
\hat{\nabla}_{AA'}\xi^B &= \nabla_{AA'}\xi^B + \Upsilon_{A'C}\epsilon_A{}^B\xi^C \\
\hat{\nabla}_{AA'}\xi_B &= \nabla_{AA'}\xi_B - \Upsilon_{A'B}\epsilon_A{}^C\xi_C = \nabla_{AA'}\xi_B - \Upsilon_{A'B}\xi_A,
\end{aligned}$$

where $\Upsilon_{AA'} = \nabla_{AA'}\ln\Omega$. The Weyl-Lanczos equation for the new metric is then given by:

$$\begin{aligned}
\hat{\Psi}_{ABCD} &= 2\hat{\epsilon}^{E'A'}\hat{\nabla}_{A'(A}\hat{L}_{BCD)E'} \\
&= 2\hat{\epsilon}^{E'A'}[\nabla_{A'(A}\hat{L}_{BCD)E'} - 3\Upsilon_{A'(A}\hat{L}_{BCD)E'} - \Upsilon_{E'(A}\hat{L}_{BCD)A'}] \\
&= 2\Omega^{-1}\epsilon^{E'A'}[\nabla_{A'(A}\hat{L}_{BCD)E'} - 2\Upsilon_{A'(A}\hat{L}_{BCD)E'}].
\end{aligned}$$

Finally, we may observe that for a spinor field X of arbitrary valence

$$\begin{aligned}
\Omega^k\nabla_{AA'}(\Omega^{-k}X) &= \nabla_{AA'}X + (-k)\Omega^k\Omega^{-k-1}(\nabla_{AA'}\Omega)X \\
&= \nabla_{AA'}X - k(\nabla_{AA'}\ln\Omega)X \\
&= \nabla_{AA'}X - k\Upsilon_{AA'}X,
\end{aligned}$$

so that the rescaled Weyl-Lanczos equation may also be written

$$\Omega^{-1}\hat{\Psi}_{ABCD} = 2\epsilon^{E'A'}\nabla_{A'(A}[\Omega^{-2}\hat{L}_{BCD)E'}]. \tag{2.16}$$

Unfortunately, the additional factor of Ω^{-1} before $\hat{\Psi}$ prevents us from establishing a direct relationship between a Lanczos potential for the original metric and one for the rescaled one.

We may however note that for a conformally flat spacetime $\hat{g}_{\mu\nu} = \Omega^2\eta_{\mu\nu}$, where $\eta_{\mu\nu}$ is the Minkowski metric, we have $\Psi = \hat{\Psi} = 0$, so that the additional factor does not matter. As such, given a Lanczos spinor L for the Minkowski metric, we know that $\hat{L}_{ABCA'} = \Omega^2 L_{ABCA'}$ will be a Lanczos spinor for $\hat{g}_{\mu\nu}$.[3]

Generalized Lanczos potential In [22] a generalized notion of the Lanczos potential is considered, where next to a symmetric spinor $\widetilde{L}_{ABCA'}$, additionally a scalar field f may be chosen, such that

$$f^{-1}\Psi_{ABCD} = 2\nabla_{(A}{}^{A'}\widetilde{L}_{BCD)A'}. \tag{2.17}$$

Comparing with equation (2.16), we can see that a generalized Lanczos potential $(\widetilde{L}_{ABCA'}, f)$ for a spacetime $g_{\mu\nu}$ yields a (non-generalized) Lanczos potential $L_{ABCA'} = f^2\widetilde{L}_{ABCA'}$ for the spacetime $f^2 g_{\mu\nu}$.

In [22] solutions for vacuum type D spacetimes are given where f is chosen to be $\Psi_2^{1/3}$ or $\Psi_2^{2/3}$. By the previous result, this corresponds to proper Lanczos potentials for the spacetimes $\Psi_2^{2/3}g_{\mu\nu}$ and $\Psi_2^{4/3}g_{\mu\nu}$, which are of course significantly different from $g_{\mu\nu}$. For this reason, the utility of considering such a generalized potential is questionable.

2.5 Newman-Penrose formulation

While the spinor formalism is very useful for investigating general properties of the Lanczos potential, the solution of the Weyl-Lanczos equations for specific spacetimes may benefit from

[3] In section 3.1 we will find that $L_{ABCA'} = \nabla_{A'(A}\chi_{BC)}$ for symmetric χ_{BC} is a Lanczos potential for conformally flat spacetimes. It is easy to verify that this may be written $L_{ABCA'} = \Omega^2\partial_{A'(A}[\Omega^{-2}\chi_{BC)}]$, which is consistent with the result obtained here.

considering a particular, geometrically chosen dyad and projecting the involved quantities onto it. This general approach, and the accompanying notation, is known as the Newman-Penrose (NP) formalism. In the following we will outline the NP formulation of the Lanczos potential, which first appeared in [23]. The derivation here follows the notation of O'Donnell [24], but does not restrict to the Lanczos differential gauge.

Given a normalized dyad $\{o^A, \iota^A\}$, the Lanczos scalars L_i are defined as follows (in some references with an extra sign):

$$
\begin{aligned}
L_0 &= L_{ABCC'} o^A o^B o^C o^{C'}, & L_4 &= L_{ABCC'} o^A o^B o^C \iota^{C'} \\
L_1 &= L_{ABCC'} o^A o^B \iota^C o^{C'}, & L_5 &= L_{ABCC'} o^A o^B \iota^C \iota^{C'} \\
L_2 &= L_{ABCC'} o^A \iota^B \iota^C o^{C'}, & L_6 &= L_{ABCC'} o^A \iota^B \iota^C \iota^{C'} \\
L_3 &= L_{ABCC'} \iota^A \iota^B \iota^C o^{C'}, & L_7 &= L_{ABCC'} \iota^A \iota^B \iota^C \iota^{C'}.
\end{aligned}
\tag{2.18}
$$

The Lanczos spinor may be reconstructed from these scalars by

$$
\begin{aligned}
L_{ABCC'} = {} & L_0 \iota_A \iota_B \iota_C \iota_{C'} - 3L_1 \iota_{(A} \iota_B o_{C)} \iota_{C'} + 3L_2 \iota_{(A} o_B o_{C)} \iota_{C'} - L_3 o_A o_B o_C \iota_{C'} \\
& - L_4 \iota_A \iota_B \iota_C o_{C'} + 3L_5 \iota_{(A} \iota_B o_{C)} o_{C'} - 3L_6 \iota_{(A} o_B o_{C)} o_{C'} + L_7 o_A o_B o_C o_{C'}.
\end{aligned}
$$

Let the symbol $\epsilon_{\mathbf{A}}{}^A$ with $\epsilon_0{}^A = o^A, \epsilon_1{}^A = \iota^A$ collectively denote the basis, and $\epsilon_A{}^{\mathbf{A}}$ the dual basis (with $\epsilon_{\mathbf{A}}{}^A \epsilon_A{}^{\mathbf{B}} = \epsilon_{\mathbf{A}}{}^{\mathbf{B}}$). Using these symbols we may now expand the Weyl-Lanczos equation in this basis. As the $\epsilon_{\mathbf{A}}{}^A$ symbols do not commute with the covariant derivative, this requires application of the Leibniz rule:

$$
\begin{aligned}
\Psi_{\mathbf{ABCD}} &= 2\epsilon_{\mathbf{A}}{}^A \epsilon_{\mathbf{B}}{}^B \epsilon_{\mathbf{C}}{}^C \epsilon_{\mathbf{D}}{}^D \nabla_{(A}{}^{E'} (L_{\mathbf{STUV'}} \epsilon_B{}^{\mathbf{S}} \epsilon_C{}^{\mathbf{T}} \epsilon_{D)}{}^{\mathbf{U}} \epsilon_{E'}{}^{\mathbf{V'}}) \\
&= 2\epsilon_{\mathbf{A}}{}^A \epsilon_{\mathbf{B}}{}^B \epsilon_{\mathbf{C}}{}^C \epsilon_{\mathbf{D}}{}^D [\epsilon_{E'}{}^{\mathbf{V'}} \epsilon_{(B}{}^{\mathbf{S}} \epsilon_C{}^{\mathbf{T}} \epsilon_D{}^{\mathbf{U}} \nabla_{A)}{}^{E'} L_{\mathbf{STUV'}} \\
&\qquad\qquad + 3L_{\mathbf{STUV'}} \epsilon_{E'}{}^{\mathbf{V'}} \epsilon_{(C}{}^{\mathbf{T}} \epsilon_D{}^{\mathbf{U}} \nabla_A{}^{E'} \epsilon_{B)}{}^{\mathbf{S}} \\
&\qquad\qquad + L_{\mathbf{STUV'}} \epsilon_{(B}{}^{\mathbf{S}} \epsilon_C{}^{\mathbf{T}} \epsilon_D{}^{\mathbf{U}} \nabla_{A)}{}^{E'} \epsilon_{E'}{}^{\mathbf{V'}}] \\
&= 2[\nabla_{(\mathbf{A}}{}^{\mathbf{V'}} L_{\mathbf{BCD})\mathbf{V'}} + 3L_{\mathbf{S}(\mathbf{CD}|\mathbf{V'}|} \epsilon_{\mathbf{B}}{}^B \nabla_{\mathbf{A})}{}^{\mathbf{V'}} \epsilon_B{}^{\mathbf{S}} + L_{(\mathbf{BCD}|\mathbf{V'}|} \epsilon_{\mathbf{E'}}{}^{E'} \nabla_{\mathbf{A})}{}^{\mathbf{E'}} \epsilon_{E'}{}^{\mathbf{V'}}] \\
&= 2[\nabla_{(\mathbf{A}}{}^{\mathbf{V'}} L_{\mathbf{BCD})\mathbf{V'}} - 3L_{\mathbf{S}(\mathbf{CD}|\mathbf{V'}|} \gamma_{\mathbf{A}}{}^{\mathbf{V'}}{}_{\mathbf{B})}{}^{\mathbf{S}} - L_{(\mathbf{BCD}|\mathbf{V'}|} \overline{\gamma}^{\mathbf{E'}}{}_{\mathbf{A})\mathbf{E'}}{}^{\mathbf{V'}}] \\
&= 2[\epsilon^{\mathbf{V'X'}} \nabla_{(\mathbf{A}|\mathbf{X'}|} L_{\mathbf{BCD})\mathbf{V'}} - 3\epsilon^{\mathbf{V'X'}} L_{\mathbf{S}(\mathbf{CD}|\mathbf{V'}|} \gamma_{\mathbf{A}|\mathbf{X'}|\mathbf{B})}{}^{\mathbf{S}} - \epsilon^{\mathbf{E'X'}} L_{(\mathbf{BCD}|\mathbf{V'}} \overline{\gamma}_{\mathbf{X'}|\mathbf{A})\mathbf{E'}}{}^{\mathbf{V'}}].
\end{aligned}
$$

In the final steps the spin coefficients $\gamma_{\mathbf{AA'B}}{}^{\mathbf{C}} = -\epsilon_{\mathbf{B}}{}^B \nabla_{\mathbf{AA'}} \epsilon_B{}^{\mathbf{C}}$ have been introduced. At this point, it is straightforward (if somewhat tedious) to explicitly evaluate this result for the relevant combinations of dyad index values 0 and 1. This yields the Weyl-Lanczos equations in the Newman-Penrose formalism:

$$
\begin{aligned}
\tfrac{1}{2}\Psi_0 &= \delta L_0 - DL_4 - (\overline{\alpha} + 3\beta - \overline{\pi})L_0 + 3\sigma L_1 + (3\epsilon - \overline{\epsilon} + \overline{\rho})L_4 - 3\kappa L_5 \\
2\Psi_1 &= 3\delta L_1 - 3DL_5 - \overline{\delta} L_4 + \Delta L_0 - (3\gamma + \overline{\gamma} + 3\mu - \overline{\mu})L_0 - 3(\overline{\alpha} + \beta - \overline{\pi} - \tau)L_1 \\
&\quad + 6\sigma L_2 + (3\alpha - \overline{\beta} + 3\pi + \overline{\tau})L_4 + 3(\epsilon - \overline{\epsilon} + \overline{\rho} - \rho)L_5 - 6\kappa L_6 \\
\Psi_2 &= \delta L_2 - DL_6 - \overline{\delta} L_5 + \Delta L_1 - \nu L_0 - (2\mu - \overline{\mu} + \gamma + \overline{\gamma})L_1 - (\overline{\alpha} - \beta - \overline{\pi} - 2\tau)L_2 \\
&\quad + \sigma L_3 + \lambda L_4 + (\alpha - \overline{\beta} + 2\pi + \overline{\tau})L_5 - (\epsilon + \overline{\epsilon} - \overline{\rho} + 2\rho)L_6 - \kappa L_7 \\
2\Psi_3 &= \delta L_3 - DL_7 - 3\overline{\delta} L_6 + 3\Delta L_2 - 6\nu L_1 + 3(\overline{\mu} - \mu + \gamma - \overline{\gamma})L_2 \\
&\quad - (\overline{\alpha} - 3\beta - 3\tau - \overline{\pi})L_3 + 6\lambda L_5 - 3(\alpha + \overline{\beta} - \overline{\tau} - \pi)L_6 - (3\epsilon + \overline{\epsilon} - \overline{\rho} + 3\rho)L_7 \\
\tfrac{1}{2}\Psi_4 &= \Delta L_3 - \overline{\delta} L_7 - 3\nu L_2 + (\overline{\mu} + 3\gamma - \overline{\gamma})L_3 + 3\lambda L_6 - (3\alpha + \overline{\beta} - \overline{\tau})L_7.
\end{aligned}
\tag{2.19}
$$

2.5.1 GHP Formulation

The Geroch-Held-Penrose (GHP) formalism [16] is an extension of the NP formalism, which uses the transformation behavior of weighted scalars to achieve a more compact notation. As the GHP formalism is less well known than the NP formalism, we will briefly summarize the concepts and notations behind it.

Given a dyad $\{o^A, \iota^A\}$, normalized such that $o_A \iota^A = 1$, the most general transformation preserving both the normalization and the flagpole null directions is

$$o^A \mapsto \lambda o^A, \quad \iota^A \mapsto \lambda^{-1} \iota^A, \tag{2.20}$$

where λ is an arbitrary (nowhere vanishing) complex field. If under this dyad transformation a scalar η transforms by

$$\eta \to \lambda^p \bar{\lambda}^q \eta,$$

then it said to be a *weighted* scalar of type $\{p, q\}$. The weighted spin coefficients have the following types:

$$\begin{array}{llll} \kappa : \{3,1\}, & \sigma : \{3,-1\}, & \rho : \{1,1\}, & \tau : \{1,-1\} \\ \kappa' : \{-3,-1\}, & \sigma' : \{-3,1\}, & \rho' : \{-1,-1\}, & \tau' : \{-1,1\}. \end{array} \tag{2.21}$$

The prime operation here effects the replacement

$$o^A \mapsto i\iota^A, \quad \iota^A \mapsto io^A, \quad o^{A'} \mapsto -i\iota^{A'}, \quad \iota^{A'} \mapsto -io^{A'}$$

and in particular we have that

$$\kappa' = -\nu, \quad \sigma' = -\lambda, \quad \rho' = -\mu, \quad \tau' = -\pi.$$

The remaining spin coefficients α, β, ϵ and γ, as well as the standard NP differential operators are not well-behaved under the transformation (2.20). However, it is possible to combine both to define a new set of weighted differential operators, which act on a $\{p, q\}$ weighted scalar η by:[4]

$$\begin{array}{ll} \text{þ}\eta = (D - p\epsilon - q\bar{\epsilon})\eta, & \text{þ}'\eta = (\Delta - p\gamma - q\overline{\gamma})\eta \\ \eth\eta = (\delta - p\beta - q\overline{\alpha})\eta, & \eth'\eta = (\overline{\delta} - p\alpha - q\overline{\beta})\eta. \end{array}$$

These operators have weights

$$\text{þ} : \{1,1\}, \quad \text{þ}' : \{-1,-1\}, \quad \eth : \{1,-1\}, \quad \eth' : \{-1,1\},$$

which combine additively with the weight of the scalar they are applied to.

We can now read off the weights of the Lanczos scalars directly from their definition in equation (2.18)

$$\begin{array}{llll} L_0 : \{3,1\}, & L_1 : \{1,1\}, & L_2 : \{-1,1\}, & L_3 : \{-3,1\}, \\ L_4 : \{3,-1\}, & L_5 : \{1,-1\}, & L_6 : \{-1,-1\}, & L_7 : \{-3,-1\}, \end{array} \tag{2.22}$$

and also determine that the prime operation acts on them by

$$L_i' = -L_{7-i}, \quad 0 \le i \le 7.$$

Furthermore we note that $\Psi_i' = \Psi_{4-i}, 0 \le i \le 4$. The NP Weyl-Lanczos equation (2.19) can now be translated into the GHP formalism by substituting the new set of differential

[4] þ is pronounced "thorn" and ð is pronounced "eth". These letters were used in Old and Middle English.

operators and making use of the primed symbols:

$$
\begin{aligned}
\tfrac{1}{2}\Psi_0 &= \eth L_0 + \text{þ} L'_3 - \overline{\tau}' L_0 + 3\sigma L_1 - \overline{\rho} L'_3 + 3\kappa L'_2 \\
2\Psi_1 &= 3\eth L_1 + 3\text{þ} L'_2 + \eth' L'_3 + \text{þ}' L_0 - (\overline{\rho}' - 3\rho')L_0 - 3(\overline{\tau} - \tau)L_1 \\
&\quad + 6\sigma L_2 + (3\tau' - \overline{\tau})L'_3 + 3(\rho - \overline{\rho})L'_2 + 6\kappa L'_1 \\
\Psi_2 &= \eth L_2 + \text{þ} L'_1 + \eth' L'_2 + \text{þ}' L_1 + \kappa' L_0 - (\overline{\rho}' - 2\rho')L_1 - (\overline{\tau}' - 2\tau)L_2 \\
&\quad + \sigma L_3 + \sigma' L'_3 + (2\tau' - \overline{\tau})L'_2 + (2\rho - \overline{\rho})L'_1 + \kappa L'_0 \\
2\Psi'_1 &= \eth L_3 + \text{þ} L'_0 + 3\eth' L'_1 + 3\text{þ}' L_2 + 6\kappa' L_1 - 3(\overline{\rho}' - \rho')L_2 - (\overline{\tau}' - 3\tau)L_3 \\
&\quad + 6\sigma' L'_2 + (3\tau' - \overline{\tau})L'_1 + (3\rho - \overline{\rho})L'_0 \\
\tfrac{1}{2}\Psi'_0 &= \text{þ}' L_3 + \eth' L'_0 + 3\kappa' L_2 - \overline{\rho}' L_3 + 3\sigma' L'_1 - \overline{\tau} L'_0.
\end{aligned}
\tag{2.23}
$$

The resulting GHP Weyl-Lanczos equations are somewhat simpler than the NP variants. Additionally, we can see that the last two equations are actually just the primed versions of the first two, so that we may, in principle, use the first three equations only and obtain the remainder by priming.

However, it should be emphasized that in the general case primed and unprimed quantities are entirely unrelated. In particular, it is not valid to take an expression for η and obtain η' simply by priming its constituent parts. For this reason it is useful to retain all five GHP Weyl-Lanczos equations, as it avoids confusion when switching from generic scalar symbols to specific values for scalars.

Chapter 3: Solutions

While the previous chapter discussed general properties of the Lanczos potential, this chapter will investigate solutions of the Weyl-Lanczos equations for particular classes of spacetimes.

The term "solutions" here may refer to two different problems: Firstly, finding *some* Lanczos spinor $L_{ABCC'}$ for a given spacetime with Weyl spinor Ψ_{ABCD}. Secondly, assuming *some* Lanczos spinor L is already known, characterizing the remaining differential gauge freedom, by finding a symmetric $M_{ABCC'}$, such that $\hat{L}_{ABCC'} = L_{ABCC'} + M_{ABCC'}$ is still a Lanczos potential of Ψ.

This is equivalent to solving $0 = \nabla_{(A}{}^{E'} M_{BCD)E'}$. While this formally resembles the Weyl-Lanczos equation for conformally flat space ($\Psi = 0$), the problems differ significantly, as the covariant derivative used here is not conformally flat in the general case.

As will be seen, both problems are somewhat orthogonal, in that in some cases we know solutions for the former but not the latter and vice-versa. Only in the case of conformally flat spacetimes both problems are solved "completely", in the sense that all degrees of freedom are accounted for.

3.1 Some algebraically special spacetimes

We will begin by considering the simple ansatz

$$L_{ABCA'} = \tfrac{1}{4}\nabla_{A'(A}\chi_{BC)}, \tag{3.1}$$

where $\chi_{AB} = \chi_{(AB)}$ is an, at this point, arbitrary symmetric spinor field. This is the spinorial equivalent of the tensor ansatz

$$L_{abc} = 2M_{ab;c} + M_{cb;a} - M_{ca;b} + g_{ca}M_b{}^k{}_{;k} - g_{cb}M_a{}^k{}_{;k}, \tag{3.2}$$

with M_{ab} antisymmetric, investigated by Chrobok [12]. Appendix A.2 establishes the relationship between the spinor and tensor versions, as well as the resulting Weyl-Lanczos equations. Inserting (3.1) into the Weyl-Lanczos equation we have

$$\Psi_{ABCD} = \tfrac{1}{2}\nabla_{(A}{}^{A'}\nabla_{|A'|B}\chi_{CD)} = -\tfrac{1}{2}\Box_{(AB}\chi_{CD)} = \Psi_{(ABC}{}^{E}\chi_{D)E}. \tag{3.3}$$

This provides a nice illustration for the simplifying power of the spinor formalism when applied to Lanczos potentials: while the equivalent reduction of the Weyl-Lanczos equation for the tensor ansatz requires a lengthy calculation, the result is immediate if the spinor formalism is employed.

Clearly, equation (3.3) is identically satisfied for conformally flat spacetimes, in which case χ_{AB} may be chosen arbitrarily. To obtain solutions for additional spacetime classes, we will switch into the Newman-Penrose formalism. Given a dyad $\{o^A, \iota^A\}$ normalized such that $o_A\iota^A = 1$, we may project the involved quantities into scalars

$$\begin{aligned}\Psi_{ABCD} &= \Psi_0\iota_A\iota_B\iota_C\iota_D - 4\Psi_1 o_{(A}\iota_B\iota_C\iota_{D)} + 6\Psi_2 o_{(A}o_B\iota_C\iota_{D)} \\ &\quad - 4\Psi_3 o_{(A}o_B o_C\iota_{D)} + \Psi_4 o_A o_B o_C o_D \\ \chi_{AB} &= \chi_0\iota_A\iota_B + 2\chi_1 o_{(A}\iota_{B)} + \chi_2 o_A o_B,\end{aligned}$$

where the Ψ_i are the Weyl scalars and $\chi_0 = \chi_{AB}o^A o^B$, $\chi_1 = \chi_{AB}o^A\iota^B$, $\chi_2 = \chi_{AB}\iota^A\iota^B$. Inserting into equation (3.3) and comparing coefficients yields the following system of equations:

$$\begin{aligned} \Psi_0 &= \quad \chi_0\Psi_1 + \ \chi_1\Psi_0 + \qquad 0 \\ -4\Psi_1 &= -3\chi_0\Psi_2 - 2\chi_1\Psi_1 + \ \chi_2\Psi_0 \\ 6\Psi_2 &= \quad 3\chi_0\Psi_3 + \qquad 0 - 3\chi_2\Psi_1 \\ -4\Psi_3 &= \ -\chi_0\Psi_4 + 2\chi_1\Psi_3 + 3\chi_2\Psi_2 \\ \Psi_4 &= \qquad\quad 0 - \ \chi_1\Psi_4 - \ \chi_2\Psi_3. \end{aligned} \tag{3.4}$$

If the spacetime is Petrov type N and we choose o^A to be a principal spinor, then only Ψ_4 is non-vanishing and the system reduces to

$$0 = -\chi_0\Psi_4, \quad \Psi_4 = -\chi_1\Psi_4.$$

This is solved by $\chi_0 = 0, \chi_1 = -1$ and χ_2 arbitrary. We may write

$$L_{ABCA'} = \nabla_{A'(A}\left[-\tfrac{1}{2}\iota_B o_{C)} + \alpha o_B o_{C)}\right], \tag{3.5}$$

where α is an arbitrary complex field and ι^A is an arbitrary spinor field satisfying $o_A\iota^A = 1$.

If the spacetime is Petrov type III and we choose o^A as a repeated principal spinor and ι^A as a singular one, then only Ψ_3 is non-vanishing and the system reduces to

$$0 = 3\chi_0\Psi_3, \quad -4\Psi_3 = 2\chi_1\Psi_3, \quad 0 = -\chi_2\Psi_3.$$

This is solved by $\chi_0 = \chi_2 = 0, \chi_1 = -2$ and we may write

$$L_{ABCA'} = -\nabla_{A'(A}(\iota_B o_{C)}). \tag{3.6}$$

Similar solutions for Petrov types D, II and I do not exist: these have canonical forms with $\Psi_1 = \Psi_3 = 0$ and $\Psi_2 \neq 0$ [26, p. 240], which is clearly incompatible with the third equation of system (3.4).

However, for Petrov type D spacetimes we can at least obtain a solution to the gauge problem $0 = \nabla_{(A}{}^{E'}M_{BCD)E'}$, using the same ansatz $M_{ABCA'} = \frac{1}{4}\nabla_{A'(A}\chi_{BC)}$. In this case the calculation is essentially the same, with the difference that the left-hand side of system (3.4) becomes zero.

If the spacetime is Petrov type D and o^A and ι^A are chosen as independent principal spinors, then only Ψ_2 is non-vanishing. The system (with left-hand side zero) reduces to

$$0 = -3\chi_0\Psi_2, \quad 0 = 3\chi_2\Psi_2,$$

which is satisfied by $\chi_0 = \chi_2 = 0$ and χ_1 arbitrary. We may write

$$M_{ABCA'} = \nabla_{A'(A}(\alpha\iota_B o_{C)}), \tag{3.7}$$

where α is an arbitrary complex field. To summarize, we have the following solutions:

Type 0: $L_{ABCA'} = \nabla_{A'(A}\chi_{BC)}$
for arbitrary symmetric χ_{AB}.

Type N: $L_{ABCA'} = \nabla_{A'(A}\left[-\frac{1}{2}\iota_B o_{C)} + \alpha o_B o_{C)}\right]$
for o_A principal, ι_B arbitrary with $o_A\iota^B = 1$, α arbitrary.

Type III: $L_{ABCA'} = -\nabla_{A'(A}(\iota_B o_{C)})$
for o_A and ι_B principal with $o_A\iota^B = 1$.

Type D: $L_{ABCA'} = \hat{L}_{ABCA'} + \nabla_{A'(A}(\alpha\iota_B o_{C)})$
for $\hat{L}_{ABCA'}$ Lanczos potential, o_A and ι_B principal with $o_A\iota^B = 1$, α arbitrary.

At this point it is interesting to consider how many degrees of freedom we have in these solution. For conformally flat space, χ_{AB} is arbitrary (apart from symmetry), so we have six (real) degrees of freedom. This exacerbates the complete differential gauge freedom.

For Petrov type N, we have two degrees of freedom in the complex field α, and another two in the choice of a normalized ι^A, for a total of four degrees of freedom. For Petrov type III, we have only a single solution, without any degrees of freedom. For Petrov type D we don't have a (general) solution, but given one, the field α would give us two additional gauge degrees of freedom.

The solutions derived here have been given by various authors in a number of different forms. The solution for conformally flat space was first given by Torres [32], using a different derivation which will be discussed in section 3.3. The gauge solution for type D spacetimes was also given by Torres [33]. The spinor forms of the type III and type N solutions (albeit without the additional field α), have first appeared in [5]. Previously, they were derived in [6] by means of the Newman-Penrose formalism, in which case the type III solution reads

$$\begin{aligned} L_0 &= \kappa, \quad L_1 = \tfrac{1}{3}\rho, \quad L_2 = -\tfrac{1}{3}\tau, \quad L_3 = -\lambda \\ L_4 &= \sigma, \quad L_5 = \tfrac{1}{3}\tau, \quad L_6 = -\tfrac{1}{3}\mu, \quad L_7 = -\nu, \end{aligned} \tag{3.8}$$

and the type N solution differs by a factor $\frac{1}{2}$ (again without the field α).

If one compares the GHP weights of the Lanczos scalars in (2.22) and the weights of the spin coefficients in (2.21), one may observe that there is always one Lanczos scalar and one spin coefficient with the same weight. The solution (3.8) assigns to each Lanczos scalar (up to proportionality) the corresponding spin coefficient with the same weight. This natural ansatz makes these solutions easy to obtain using the GHP formalism, where they were first given in [11].

Conformally flat spacetimes The solution for conformally flat spacetimes is particularly interesting, because the obtained solution has the full set of differential gauge degrees. As such, it should always be possible to choose the field χ_{AB}, such that the potential is in a certain differential gauge F_{AB}. We may write this condition as

$$F_{AB} = \nabla^{CC'}\nabla_{C'(A}\chi_{BC)} = \tfrac{1}{3}(\Box\chi_{AB} + 2\nabla_{C'C}\nabla_{(A}{}^{C'}\chi_{B)}{}^{C}),$$

where

$$\begin{aligned} \nabla_{C'C}\nabla_{A}{}^{C'}\chi_{B}{}^{C} &= \nabla_{C'[C}\nabla_{A]}{}^{C'}\chi_{B}{}^{C} + \nabla_{C'(C}\nabla_{A)}{}^{C'}\chi_{B}{}^{C} \\ &= \tfrac{1}{2}\Box\chi_{BA} - 3\Lambda\chi_{BA} + X_{CABE}\chi^{EC} \end{aligned}$$

and

$$X_{C(AB)E}\chi^{EC} = \Psi_{EABC}\chi^{EC} - \Lambda\chi_{AB}.$$

Of course, in the conformally flat case we have $\Psi = 0$, so that the overall condition reduces to

$$F_{AB} = \tfrac{2}{3}(\Box - 4\Lambda)\chi_{AB} = \tfrac{2}{3}(\Box - \tfrac{1}{6}R)\chi_{AB}.$$

For the Lanczos differential gauge $F_{AB} = 0$ in particular we have $(\Box - \frac{1}{6}R)\chi_{AB} = 0$. This wave equation looks similar to the conformally invariant scalar wave equation $(\Box + \frac{1}{6}R)\phi = 0$, but unfortunately differs in sign.

Grimminger [17] investigated solutions to the tensorial equivalent $(\Box - \frac{1}{6}R)M_{ab} = 0$ of this equation. For $g_{ab} = \Omega^2\eta_{ab}$ it was found that $M_{ab} = \Omega^3 A_{ab}$ with A_{ab} antisymmetric and constant ($A_{ab,c} = 0$) is a solution. However, the Lanczos potential generated from this choice of M_{ab} reduces to $L_{abc} = 0$. As such, no non-trivial Lanczos potentials for conformally flat space-times satisfying the Lanczos differential gauge are currently known.

3.2 Kerr-Schild spacetimes

A Lanczos potential for Kerr spacetimes has been first given by Bergqvist [9]. With $\{o^A, \iota^A\}$ referring to a normalized dyad of principal spinors (which exists, because the spacetime is type D), the Lanczos potential may be written in the form (2.14) with

$$L_{ABCA'} = \nabla_{(A}{}^{B'} H_{BC)A'B'}, \quad H_{ABA'B'} = f o_A o_B o_{A'} o_{B'},$$

where

$$f = \frac{\rho + \bar{\rho}}{4\rho^3}\Psi_2 = \frac{\rho + \bar{\rho}}{4} M \tag{3.9}$$

depends on the mass parameter M and the spin coefficient ρ. This Lanczos potential has a special property: If we consider

$$\Gamma_{ABCA'} = \nabla_{(A}{}^{B'} H_{B)CA'B'},$$

such that $L_{ABCA'} = \Gamma_{(ABC)A'}$, but Γ not necessarily symmetric in BC, then the connection defined by

$$\hat{\nabla}_{AA'}\xi_B = \nabla_{AA'}\xi_B - 2\Gamma^E{}_{BAA'}\xi_E \tag{3.10}$$

is flat, i.e., the curvature tensor for this connection vanishes identically. Due to symmetry of Γ on the first two indices, we have that

$$\hat{\nabla}_{AA'}\epsilon_{BC} = -2\Gamma^E{}_{BAA'}\epsilon_{EC} - 2\Gamma^E{}_{CAA'}\epsilon_{BE} = -2\Gamma_{CBAA'} + 2\Gamma_{BCAA'} = 0,$$

such that the connection $\hat{\nabla}$ is metric, but not torsion-free.

Bergqvist starts with this flat connection construction, which was already previously known [10], and shows that it also acts as a Lanczos potential using a relatively lengthy proof in the GHP formalism. We will instead refer to a later proof by Andersson and Edgar [3], which is both simpler and applies to a wider range of Kerr-Schild spacetimes.

Andersson and Edgar begin by considering the effect of the choice of the connection in equation (3.10) on the curvature. As the connection is no longer torsion-free, we are now in a Riemann-Cartan space, where the spinor decomposition of the curvature

$$\begin{aligned}\hat{R}_{abcd} = {} & [\hat{\Psi}_{ABCD} + 2(\epsilon_{B(C}\hat{\Sigma}_{D)A} + \epsilon_{A(C}\hat{\Sigma}_{D)B}) + \hat{\Lambda}(\epsilon_{AD}\epsilon_{BC} + \epsilon_{AC}\epsilon_{BD})]\epsilon_{A'B'}\epsilon_{C'D'} \\ & + \hat{\Phi}_{ABC'D'}\epsilon_{A'B'}\epsilon_{CD} + c.c.\end{aligned}$$

has an additional component $\hat{\Sigma}_{AB}$, and $\hat{\Phi}_{ABA'B'}$ is no longer necessarily real. The curvature components of $\hat{\nabla}$ relate to those of the Levi-Civita connection by:

$$\begin{aligned}\hat{\Psi}_{ABCD} &= \Psi_{ABCD} - 2\nabla_{(A}{}^{E'}\Gamma_{BCD)E'} - 4\Gamma_{E(AB}{}^{E'}\Gamma^E{}_{CD)E'}, \\ 3\hat{\Lambda} &= 3\Lambda - \nabla_A{}^{E'}\Gamma^{AB}{}_{BE'} - \Gamma_{EABF'}\Gamma^{BEAF'} + \Gamma^B{}_{EBF'}\Gamma^{EA}{}_A{}^{F'} \\ 4\hat{\Sigma}_{AD} &= \nabla^{BE'}\Gamma_{B(AD)E'} - \nabla_{(A}{}^{E'}\Gamma^B{}_{D)BE'} - 2\Gamma_{E(D|B|}{}^{F'}\Gamma^{BE}{}_{A)F'} - 2\Gamma_{E(AD)}{}^{F'}\Gamma^{BE}{}_{BF'} \\ \hat{\Phi}_{ABC'D'} &= \Phi_{ABC'D'} + 2\nabla_{(B|E'}\overline{\Gamma}_{C'D'}{}^{E'}{}_{|A)} + 4\overline{\Gamma}_{E'D'F'(A|}\overline{\Gamma}_{C'}{}^{E'F'}{}_{|B)}.\end{aligned}$$

We see that if $\Gamma_{(ABC)A'}$ is a Lanczos potential of Ψ_{ABCD}, then the first two terms of $\hat{\Psi}_{ABCD}$ cancel. The following lemma follows immediately:

Lemma 1 (Andersson and Edgar 1998) *Given an asymmetric metric connection $\hat{\nabla}_{AA'}$ of the form (3.10), any two of the following statements imply the third:*

1. $\hat{\Psi}_{ABCD} = 0$*, i.e., the Riemann-Cartan space is conformally flat.*
2. $\Gamma_{E(AB}{}^{E'}\Gamma^E{}_{CD)E'} = 0.$

3. $L_{ABCA'} = \Gamma_{(ABC)A'}$ *is a Lanczos potential of* Ψ_{ABCD}.

Returning to the original problem, a Kerr-Schild spacetime has the general form

$$g_{ab} = \eta_{ab} + 2fl_a l_b,$$

where $l_a = o_A o_{A'}$ is a null vector field and f a real scalar field. For the case of the Kerr spacetime in particular, f is given by equation (3.9) and l_a is aligned to a geodesic, shear-free null congruence. The asymmetric metric connection defined by

$$\Gamma_{ABCA'} = \nabla_{(A}{}^{B'} H_{B)CA'B'}, \quad H_{ABA'B'} = f o_A o_B o_{A'} o_{B'}$$

has been shown to be flat for the Kerr spacetime in [10], and for the general class of Kerr-Schild spacetimes in [18]. This satisfies the first condition of the above lemma, so it remains to be shown that $\Gamma_{E(AB}{}^{E'}\Gamma^{E}{}_{CD)E'} = 0$. We have that

$$\begin{aligned}\Gamma_{ABCA'}o^{A'} &= \nabla_{(A}{}^{B'}[f o_{B)} o_C o_{A'} o_{B'} o^{A'}] - f o_{(A|} o_C o_{A'} o_{B'} \nabla_{|B)}{}^{B'} o^{A'} \\ &= -f o_{(A|} o_C o^{A'} o^{B'} \nabla_{|B)B'} o_{A'}.\end{aligned}$$

If o^A is aligned to a geodesic, shear-free null congruence, then $o^{A'} o^{B'} \nabla_{BB'} o_{A'} = 0$. Thus we have $\Gamma_{ABCA'} o^{A'} = 0$ and consequently there must be some M_{ABC} such that $\Gamma_{ABCA'} = M_{ABC} o_{A'}$. The desired property then follows directly. By the preceding lemma, we now have that

$$L_{ABCA'} = \Gamma_{(ABC)A'} = \nabla_{(A}{}^{B'}[f o_B o_{C)} o_{A'} o_{B'}]$$

is a Lanczos potential for any Kerr-Schild spacetime where l_a is aligned to a geodesic, shear-free null congruence. This in particular also includes the Kerr spacetime.

3.3 Spacetimes with geodesic, shear-free null congruence

There are two generalizations of the Kerr-Schild result to a wider class of spacetimes, one by Andersson [2] and another by Torres [32]. Interestingly, the solution by Torres predates both the Kerr-Schild solution and the original Kerr solution due to Bergqvist [9].

In both cases, spacetimes that admit a geodesic, shear-free null congruence $l_a = o_A o_{A'}$ are considered, under the additional restriction that the Ricci spinor is aligned to this congruence by $\Phi_{ABA'B'} o^A o^B = 0$. In terms of NP quantities this means $\kappa = \sigma = 0$ and $\Phi_{00} = \Phi_{01} = \Phi_{02} = 0$. By the Goldberg-Sachs theorem it follows that $\Psi_0 = \Psi_1 = 0$, so the spacetime must be algebraically special.

Andersson [2] additionally requires that the null congruence be expanding ($\rho \neq 0$) and the scalar curvature be constant ($\Lambda = const$). Andersson uses the GHP formalism and the method of ρ-integration [1] to approach this problem in multiple steps:

First, Lanczos potentials of the form $L_{ABCA'} = M_{ABC} o_{A'}$ are considered. Then, this is restricted to $L_{ABCA'} = \nabla_{(A}{}^{B'} H_{BC)A'B'}$, where $H_{ABA'B'} = Q_{AB} o_{A'} o_{B'}$ with $Q_{AB} = Q_{(AB)}$. In both cases the Weyl-Lanczos equations are integrated, conditions for the functions of integration are established and it is shown that these can always be satisfied.

Finally, it is considered under what circumstances it is possible to choose Q_{AB}, such that the asymmetric metric connection defined by $\Gamma_{ABCA'} = \nabla_{(A}{}^{B'} H_{B)CA'B'}$, as in the previous section, has vanishing curvature components. From the preceding Lemma we already know that $\hat{\Psi}_{ABCD} = 0$ must hold. Andersson additionally shows that $\hat{\Sigma}_{AB} = 0$. Lastly, it is shown that a choice of Q_{AB} such that also $\hat{\Lambda} = 0$ and $\hat{\Phi}_{ABA'B'} = 0$ (making the connection completely flat) exists if and only if $\Lambda = 0$.

While Torres [32] investigates a similar class of spacetimes (without the restrictions $\rho \neq 0$ and $\Lambda = const$) and the same form of potentials, his approach is based on methods from the theory of null strings and $\mathcal{H}$-spaces in complex general relativity. As this goes beyond what can be briefly summarized here, we will only outline the general approach and cite the final result. We note that in complex spacetimes, objects that were previously related by complex conjugation split into independent objects, for example Ψ and $\overline{\Psi}$ become the independent quantities Ψ and $\widetilde{\Psi}$.

If the complex extension of the spacetime is considered, then it is known (for the given class of spacetimes), that there exists a congruence of totally null 2-surfaces (all tangent vectors of which are null), called a null string [30]. This null string induces a natural spinor structure [15], which, given a specific choice of null tetrad, allows the metric and curvature to be expressed in a certain form. Torres reformulates this in a covariant representation [31], whereby the metric may be written

$$ds^2 = {}^{\circ}ds^2 + \phi^{-2}\Omega_{A'B'}o_C o_D g^{CA'}g^{DB'},$$

where $ds^2 = 2\phi^{-2}dq^{A'}dp_{A'}$ is conformally flat, with ϕ being a complex field and $q^{A'}, p^{A'}$ being complex coordinates. $\Omega_{A'B'}$ is symmetric and $g^{AB'}$ refers to a certain cotangent null tetrad (in terms of which $ds^2 = -\frac{1}{2}g_{AB'}g^{AB'}$). Given this representation, the Lanczos potential $\widetilde{L}$ of $\widetilde{\Psi}$ is given by

$$\widetilde{L}_{A'B'C'R} = \tfrac{1}{2}\nabla_{(A'}{}^{S}[\phi^{-2}\Omega_{B'C')}o_R o_S],$$

which is of the same form as considered by Andersson. As a special case of this result, Torres also gives the Lanczos potential for conformally flat spacetimes, which was already discussed in section 3.1.

Chapter 4: Discussion

The topic of this thesis was the spinorial formulation of the theory of Lanczos potentials. We have seen that many results can be more easily obtained and more compactly expressed in the spinor formalism. The reason for this simplification is presumably that the Weyl spinor possesses much simpler symmetry properties than the Weyl tensor.

We have discussed a number of properties of the Lanczos potential, the most important of which is the Illge wave equation, which plays an integral part in proving existence and uniqueness of the Lanczos potential. The formulation in the related NP and GHP formalism was also considered. Finally, a number of spinorial solutions, which are applicable to a relatively wide class of spacetimes, have been discussed.

Of course, the question that is ultimately most interesting, is to what degree the use of Lanczos potentials (be it using spinors or tensors) is useful in solving gravitational problems and whether these potentials hold any deeper physical significance.

In this context, the Lanczos potential is commonly seen in analogy with the electromagnetic (EM) potential [28]: The EM tensor may be written $F_{ab} = \nabla_{[a}A_{b]}$, where A_b is the EM potential, and satisfies the field equation $\nabla^a F_{ab} = J_b$, where J_b is the source current. Similarly, the Weyl tensor may be written $C_{abcd} = f(\nabla_e L_{fgh})$, and satisfies the field equation $\nabla^a C_{abcd} = J_{bcd}$ (Bianchi identity), where J_{bcd} is constructed from derivatives of the Ricci tensor. The Lanczos differential gauge $\nabla^c L_{abc} = 0$ may be seen as the analogon of the Lorentz gauge $\nabla^a A_a = 0$. In vacuum and Lorentz gauge, the EM wave equation reduces to $\Box A_a = 0$. Similarly, in vacuum and Lanczos differential gauge the Illge wave equation is $\Box L_{abc} = 0$.

Clearly, there are many similarities between the EM potential and the Lanczos potential. But while the EM potential has proven to be decidedly useful for solving EM problems and, through the Aharonov-Bohm effect, has acquired physical significance beyond a mere mathematical tool, the same cannot be said of the Lanczos potential. Most publications in this field deal with finding Lanczos potentials for specific spacetimes, while useful applications are few and far between. To give two examples:

For the Kerr solution discussed in section 3.2, Bergqvist expresses a quasilocal momentum in terms of the Lanczos potential [9]. As this construction depends on the flat asymmetric metric connection from which the Lanczos potential is induced, and has been given in terms of that connection previously [10], the role of the Lanczos potential itself is somewhat tenuous.

Roberts [28] investigates whether the Lanczos potential may exhibit something similar to the Ahoronov-Bohm effect in a quantum gravitational setting. However, the result is inconclusive. Of course, other attempts at interpretation exist, but nothing stands out as particularly convincing.

A significant roadblock to the practical applicability of the Lanczos potential is probably the circumstance that no general approach to finding a Lanczos potential for a given spacetime is known (and the same holds true for enforcing the Lanczos differential gauge). Solutions for many specific spacetimes and spacetime classes are known, but derivations use a wide range of different methods, which are specific to the considered geometry and not unifiable in any obvious manner.

While Lanczos' original derivation based on variational calculus seems to imply a deep physical significance of the Lanczos potential, Bampi and Caviglia first showed that the

Lanczos potential is not connected to the role of the Weyl tensor as a curvature component, but rather is a result of its algebraic symmetries. Illge later showed that the Lanczos potential is only one of rather general class of spinor potentials.

Especially in light of this, it is my personal suspicion that the Lanczos potential is more of a mathematical curiosity, than a fundamental object with a profound physical significance.

Bibliography

[1] F. Andersson. Integration of the GHP equations in spacetimes admitting a geodesic shear-free expanding null congruence, 2000. arXiv:gr-qc/0001092.

[2] F. Andersson. Lanczos potentials and curvature-free connections aligned to a geodesic shear-free expanding null congruence, 2000. arXiv:gr-qc/0002016.

[3] F. Andersson and S. B. Edgar. Curvature-free asymmetric metric connections and Lanczos potentials in Kerr–Schild spacetimes. *Journal of Mathematical Physics*, 39(5):2859–2861, 1998.

[4] F. Andersson and S. B. Edgar. Local existence of spinor- and tensor potentials, 2000. arXiv:gr-qc/9902080v2.

[5] F. Andersson and S. B. Edgar. Spin coefficients as Lanczos scalars: Underlying spinor relations. *Journal of Mathematical Physics*, 41(5):2990–3001, 2000.

[6] G. Ares de Parga, O. Chavoya, and J. L. López-Bonilla. Lanczos potential. *Journal of Mathematical Physics*, 30(6):1294–1295, 1989.

[7] F. Bampi and G. Caviglia. Third-order tensor potentials for the Riemann and Weyl tensors. *General Relativity and Gravitation*, 15(4):375–386, 1983.

[8] P. Bell and P. Szekeres. Some properties of higher spin rest-mass zero fields in general relativity. *International Journal of Theoretical Physics*, 6(2):111–121, 1972.

[9] G. Bergqvist. A Lanczos potential in Kerr geometry. *Journal of Mathematical Physics*, 38(6):3142–3154, 1997.

[10] G. Bergqvist and M. Ludvigsen. Quasilocal momentum and angular momentum in Kerr spacetime. *Classical and Quantum Gravity*, 8(4):697, 1991.

[11] J. Caltenco, J. López-Bonilla, and A. Zúñiga-Segundo. Lanczos spintensor and GHP formalism. *Czechoslovak Journal of Physics*, 52(8):901–909, 2002.

[12] T. Chrobok. Private communication, 2017.

[13] S. Edgar. The wave equations for the Lanczos tensor/spinor, and a new tensor identity. *Modern Physics Letters A*, 09(06):479–482, 1994.

[14] S. Edgar and A. Höglund. The Lanczos potential for the Weyl curvature tensor: existence, wave equation and algorithms. *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 453(1959):835–851, 1997.

[15] J. D. Finley III and J. F. Plebański. The intrinsic spinorial structure of hyperheavens. *Journal of Mathematical Physics*, 17(12):2207–2214, 1976.

[16] R. Geroch, A. Held, and R. Penrose. A space-time calculus based on pairs of null directions. *Journal of Mathematical Physics*, 14(7):874–881, 1973.

[17] J. Grimminger. On Lanczos potentials of conformally flat space-times. Bachelor's thesis, TU Berlin, 2017.

[18] G. Harnett. The flat generalized affine connection and twistors for the Kerr solution. *Classical and Quantum Gravity*, 10(2):407, 1993.

[19] R. Illge. On potentials for several classes of spinor and tensor fields in curved spacetimes. *General Relativity and Gravitation*, 20(6):551–564, 1988.

[20] C. Lanczos. A remarkable property of the Riemann-Christoffel tensor in four dimensions. *Annals of Mathematics*, 39(4):842–850, 1938.

[21] C. Lanczos. The splitting of the Riemann tensor. *Rev. Mod. Phys.*, 34:379–389, Jul 1962.

[22] J. L. Lopez-Bonilla, J. Morales, D. Naverrete, and M. A. Rosales. Lanczos spin tensor for empty type D spacetimes. *Classical and Quantum Gravity*, 10(10):2153, 1993.

[23] W. F. Maher and J. D. Zund. A spinor approach to the Lanczos spin tensor. *Il Nuovo Cimento A (1965-1970)*, 57(4):638–648, 1968.

[24] P. O'Donnell. *Introduction to 2-spinors in General Relativity.* World Scientific, 2003.

[25] R. Penrose and W. Rindler. *Spinors and Space-Time: Volume 1, Two-Spinor Calculus and Relativistic Fields.* Cambridge Monographs on Mathematical Physics. Cambridge University Press, 1987.

[26] R. Penrose and W. Rindler. *Spinors and Space-Time: Volume 2, Spinor and Twistor Methods in Space-Time Geometry.* Cambridge Monographs on Mathematical Physics. Cambridge University Press, 1988.

[27] M. D. Roberts. The double duals and gauges of the Lanczos tensor. *Czechoslovak Journal of Physics*, 42(2):151–158, 1992.

[28] M. D. Roberts. The physical interpretation of the Lanczos tensor. *Il Nuovo Cimento B (1971-1996)*, 110(10):1165–1176, 1995.

[29] A. Taub. Lanczos' splitting of the Riemann tensor. *Computers & Mathematics with Applications*, 1(3):377 – 380, 1975.

[30] G. F. Torres del Castillo. Null strings and Bianchi identities. *Journal of Mathematical Physics*, 24(3):590–596, 1983.

[31] G. F. Torres del Castillo. Null strings and hertz potentials. *Journal of Mathematical Physics*, 25(2):342–346, 1984.

[32] G. F. Torres del Castillo. Lanczos potential for algebraically special spacetimes. *Rev. Mex. de Fisica*, 37:246–254, 1991.

[33] G. F. Torres del Castillo. Lanczos potential for some type D space–times. *Journal of Mathematical Physics*, 36(1):195–200, 1995.

[34] R. M. Wald. *General relativity.* University of Chicago press, 1984.

Appendix A: Proofs

A.1 Alternative derivation of spinorial Weyl-Lanczos equation

The equivalence of the tensorial Weyl-Lanczos equation (2.1) and the spinorial equation $\Psi_{ABCD} = 2\nabla^{E'}{}_{(A}L_{BCD)E'}$ was shown in section 2.2 based on a simple symmetry argument. Here a proof through explicit algebraic manipulation will be given instead. This approach is much more tedious (at least if the Lanczos algebraic gauge is not imposed), but serves to illustrate the somewhat non-trivial way in which the tensorial symmetries translate into the spinorial ones.

We will start from the Weyl-Lanczos equation already transvected with $\epsilon^{A'B'}\epsilon^{C'D'}$, as given in equation (2.6):

$$\begin{aligned} 4\Psi_{ABCD} = {} & 2\nabla_D{}^{C'}L_{ABCC'} + 2\nabla_C{}^{D'}L_{ABDD'} + 2\nabla_B{}^{A'}L_{CDAA'} + 2\nabla_A{}^{B'}L_{CDBB'} \\ & - \tfrac{1}{2}\epsilon_{AC}\epsilon^{B'D'}(L_{BB'DD'} + L_{DD'BB'}) - \tfrac{1}{2}\epsilon_{AD}\epsilon^{B'C'}(L_{BB'CC'} + L_{CC'BB'}) \\ & - \tfrac{1}{2}\epsilon_{BD}\epsilon^{A'C'}(L_{AA'CC'} + L_{CC'AA'}) - \tfrac{1}{2}\epsilon_{BC}\epsilon^{A'D'}(L_{AA'DD'} + L_{DD'AA'}) \\ & - \tfrac{4}{3}\nabla_{LL'}L^{KK'LL'}{}_{KK'}(\epsilon_{AC}\epsilon_{DB} + \epsilon_{AD}\epsilon_{CB}). \end{aligned} \tag{A.1}$$

We remind that we have defined $L_{AA'} := L_{AB}{}^{B}{}_{A'}$ and due to the cyclic property $L_{AA'} = \overline{L}_{AA'}$. Furthermore we use

$$L_{ab} := L_a{}^k{}_{b;k} - L_a{}^k{}_{k;b}.$$

In spinor form the latter corresponds to

$$\begin{aligned} L_{AA'BB'} &= \nabla_{EE'}L_{AA'}{}^{EE'}{}_{BB'} - \nabla_{BB'}L_{AA'}{}^{EE'}{}_{EE'} \\ &= \nabla_{EE'}(L_A{}^E{}_{BB'}\epsilon_{A'}{}^{E'} + \overline{L}_{A'}{}^{E'}{}_{B'B}\epsilon_A{}^E) - \nabla_{BB'}(L_A{}^E{}_{EE'}\epsilon_{A'}{}^{E'} + \overline{L}_{A'}{}^{E'}{}_{E'E}\epsilon_A{}^E) \\ &= \nabla_{EE'}(L_A{}^E{}_{BB'}\epsilon_{A'}{}^{E'} + \overline{L}_{A'}{}^{E'}{}_{B'B}\epsilon_A{}^E) + 2\nabla_{BB'}L_{AA'}, \end{aligned}$$

where in the last line the cyclic property has been used. We observe that

$$\begin{aligned} L_{AA'B}{}^{A'} &= \nabla_{EE'}(L_A{}^E{}_B{}^{A'}\epsilon_{A'}{}^{E'} + \overline{L}_{A'}{}^{E'A'}{}_B\epsilon_A{}^E) + 2\nabla_B{}^{A'}L_{AA'} \\ &= \nabla_{EE'}(L_A{}^E{}_B{}^{E'} + L_B{}^{E'}\epsilon_A{}^E) + 2\nabla_B{}^{A'}L_{AA'} \\ &= \nabla_{EE'}L_A{}^E{}_B{}^{E'} + \nabla_{AE'}L_B{}^{E'} - 2\nabla_{BE'}L_A{}^{E'}, \end{aligned}$$

where again the cyclic property has been used. We now consider the following expression:

$$\begin{aligned}(L_{AA'BB'} + L_{BB'AA'})\epsilon^{A'B'} &= L_{AA'B}{}^{A'} - L_{BA'A}{}^{A'} \\ &= \nabla_{EE'} L_A{}^E{}_B{}^{E'} + \nabla_{AE'} L_B{}^{E'} - 2\nabla_{BE'} L_A{}^{E'} \\ &\quad - \nabla_{EE'} L_B{}^E{}_A{}^{E'} - \nabla_{BE'} L_A{}^{E'} + 2\nabla_{AE'} L_B{}^{E'} \\ &= \nabla_{EE'} L_A{}^E{}_B{}^{E'} - \nabla_{EE'} L_B{}^E{}_A{}^{E'} + 3\nabla_{AE'} L_B{}^{E'} - 3\nabla_{BE'} L_A{}^{E'} \\ &= 2\nabla_{EE'} L_{[A}{}^E{}_{B]}{}^{E'} + 6\nabla_{[A|E'|} L_{B]}{}^{E'} \\ &= \nabla_{EE'} L_D{}^{EDE'} \epsilon_{AB} + 3\nabla_{DE'} L^{DE'} \epsilon_{AB} \\ &= 4\nabla_{EE'} L^{EE'} \epsilon_{AB}.\end{aligned}$$

For the last term of the Weyl-Lanczos equation we have

$$\begin{aligned}\nabla_{LL'} L^{KK'LL'}{}_{KK'} &= \nabla_{LL'} (L^{KL}{}_{KK'} \epsilon^{K'L'} + \overline{L}^{K'L'}{}_{K'K} \epsilon^{KL}) \\ &= 2\nabla_{LL'} L^{LL'},\end{aligned}$$

again exploiting the cyclic property. Inserting the obtained expressions into (A.1) we get

$$\begin{aligned}4\Psi_{ABCD} &= 2\nabla_D{}^{C'} L_{ABCC'} + 2\nabla_C{}^{D'} L_{ABDD'} + 2\nabla_B{}^{A'} L_{CDAA'} + 2\nabla_A{}^{B'} L_{CDBB'} \\ &\quad - 2\epsilon_{AC} \nabla_{EE'} L^{EE'} \epsilon_{BD} - 2\epsilon_{AD} \nabla_{EE'} L^{EE'} \epsilon_{BC} \\ &\quad - 2\epsilon_{BD} \nabla_{EE'} L^{EE'} \epsilon_{AC} - 2\epsilon_{BC} \nabla_{EE'} L^{EE'} \epsilon_{AD} \\ &\quad - \tfrac{2}{3} \nabla_{EE'} L^{EE'} (\epsilon_{AC}\epsilon_{DB} + \epsilon_{AD}\epsilon_{CB}) \\ &= 2\nabla_D{}^{C'} L_{ABCC'} + 2\nabla_C{}^{D'} L_{ABDD'} + 2\nabla_B{}^{A'} L_{CDAA'} + 2\nabla_A{}^{B'} L_{CDBB'} \\ &\quad + \tfrac{4}{3} \nabla_{EE'} L^{EE'} (\epsilon_{AC}\epsilon_{DB} + \epsilon_{AD}\epsilon_{CB})\end{aligned}$$

and consequently

$$\begin{aligned}\Psi_{ABCD} &= \tfrac{1}{2} (\nabla_D{}^{E'} L_{ABCE'} + \nabla_C{}^{E'} L_{ABDE'} + \nabla_B{}^{E'} L_{CDAE'} + \nabla_A{}^{E'} L_{CDBE'}) \\ &\quad + \tfrac{2}{3} \nabla_{EE'} L^{EE'} \epsilon_{A(C}\epsilon_{D)B}.\end{aligned} \tag{A.2}$$

Defining $N_{ABCD} := \nabla_A{}^{E'} L_{BCDE'}$ as a shorthand, the last term may be expanded in the following manner:

$$\begin{aligned}\nabla_{EE'} L^{EE'} \epsilon_{A(C}\epsilon_{D)B} &= -\tfrac{1}{4} N_E{}^E{}_F{}^F (\epsilon_{AC}\epsilon_{DB} + \epsilon_{DB}\epsilon_{AC} + \epsilon_{AD}\epsilon_{CB} + \epsilon_{CB}\epsilon_{AD}) \\ &= -(N_{[AC][DB]} + N_{[DB][AC]} + N_{[AD][CB]} + N_{[CB][AD]}) \\ &= -\tfrac{1}{4} (N_{ACDB} - N_{ACBD} - N_{CADB} + N_{CABD} \\ &\qquad + N_{DBAC} - N_{DBCA} - N_{BDAC} + N_{BDCA} \\ &\qquad + N_{ADCB} - N_{ADBC} - N_{DACB} + N_{DABC} \\ &\qquad + N_{CBAD} - N_{CBDA} - N_{BCAD} + N_{BCDA}) \\ &= -\tfrac{1}{4} (2N_{ACDB} + 2N_{CABD} + 2N_{DABC} + 2N_{BCDA} \\ &\qquad - N_{ABCD} - N_{CADB} - N_{DBCA} - N_{BADC} \\ &\qquad - N_{ABDC} - N_{DACB} - N_{CBDA} - N_{BACD}).\end{aligned}$$

Inserting this back into (A.2), we find that these terms combine precisely so as to form a full symmetrization operation:

$$\Psi_{ABCD} = \tfrac{1}{6} \cdot 12 \nabla_{(A}{}^{E'} L_{BCD)E'} = 2\nabla_{(A}{}^{E'} L_{BCD)E'}.$$

Thus we arrive at the usual spinorial Weyl-Lanczos equation.

A.2 Relation between spinor and tensor ansatz for some algebraically special spacetimes

Section 3.1 discusses solutions obtained from the ansatz $L_{ABCA'} = \frac{1}{4}\nabla_{A'(A}\chi_{BC)}$, which is based on a tensor ansatz previously investigated by Chrobok [12]. We will now establish how these are related and also compare the obtained expressions for the Weyl-Lanczos equation.

We start from the tensor ansatz

$$L_{abc} = 2M_{ab;c} + M_{cb;a} - M_{ca;b} + g_{ca}M_b{}^k{}_{;k} - g_{cb}M_a{}^k{}_{;k}, \tag{A.3}$$

where M_{ab} is antisymmetric. It is readily verified that this satisfies the symmetry properties and is in Lanczos algebraic gauge. Due to antisymmetry (and reality), the spinor form of M_{ab} decomposes as

$$M_{AA'BB'} = \chi_{AB}\epsilon_{A'B'} + \overline{\chi}_{A'B'}\epsilon_{AB},$$

where $\chi_{AB} = \chi_{(AB)}$. Writing equation (A.3) in spinor components gives

$$\begin{aligned} L_{ABCC'}\epsilon_{A'B'} + \overline{L}_{A'B'C'C}\epsilon_{AB} = {} & 2\nabla_{CC'}(\chi_{AB}\epsilon_{A'B'} + \overline{\chi}_{A'B'}\epsilon_{AB}) \\ & + \nabla_{AA'}(\chi_{CB}\epsilon_{C'B'} + \overline{\chi}_{C'B'}\epsilon_{CB}) \\ & - \nabla_{BB'}(\chi_{CA}\epsilon_{C'A'} + \overline{\chi}_{C'A'}\epsilon_{CA}) \\ & + \epsilon_{CA}\epsilon_{C'A'}\nabla_{KK'}(\chi_B{}^K\epsilon_{B'}{}^{K'} + \overline{\chi}_{B'}{}^{K'}\epsilon_B{}^K) \\ & - \epsilon_{CB}\epsilon_{C'B'}\nabla_{KK'}(\chi_A{}^K\epsilon_{A'}{}^{K'} + \overline{\chi}_{A'}{}^{K'}\epsilon_A{}^K) \end{aligned}$$

and transvection with $\epsilon^{A'B'}$ yields

$$\begin{aligned} 2L_{ABCC'} = {} & 4\nabla_{CC'}\chi_{AB} + \nabla_{AC'}\chi_{CB} + \nabla_{BC'}\chi_{CA} \\ & + \nabla_{AA'}\overline{\chi}_{C'}{}^{A'}\epsilon_{CB} + \nabla_{BB'}\overline{\chi}_{C'}{}^{B'}\epsilon_{CA} \\ & - \epsilon_{CA}\nabla_{KC'}\chi_B{}^K - \epsilon_{CA}\nabla_{BK'}\overline{\chi}_{C'}{}^{K'} - \epsilon_{CB}\nabla_{KC'}\chi_A{}^K - \epsilon_{CB}\nabla_{AK'}\overline{\chi}_{C'}{}^{K'}. \end{aligned} \tag{A.4}$$

As the ansatz (A.3) is in Lanczos algebraic gauge, we must have $L_{ABCC'} = L_{(ABC)C'}$. Applying the symmetrization to (A.4), all the terms containing ϵ_{AB} vanish and the first three terms combine into

$$L_{ABCC'} = 3\nabla_{C'(A}\chi_{BC)}.$$

This matches the spinor ansatz up to a normalization factor of 1/12.

After a lengthy calculation, Chrobok reduces the Weyl-Lanczos equation for the ansatz (A.3) to

$$C_{abcd} = 3(C^k{}_{bdc}M_{ka} - C^k{}_{adc}M_{kb} + C^k{}_{dba}M_{kc} - C^k{}_{cba}M_{kd}). \tag{A.5}$$

Writing the first term in spinor components and transvecting with $\epsilon^{A'B'}\epsilon^{C'D'}$, we have

$$\begin{aligned} C^k{}_{bdc}M_{ka}\epsilon^{A'B'}\epsilon^{C'D'} &= (\Psi^K{}_{BDC}\epsilon^{K'}{}_{B'}\epsilon_{D'C'} + \overline{\Psi}^{K'}{}_{B'D'C'}\epsilon^K{}_B\epsilon_{DC}) \\ &\qquad \cdot (\chi_{KA}\epsilon_{K'A'} + \overline{\chi}_{K'A'}\epsilon_{KA})\epsilon^{A'B'}\epsilon^{C'D'} \\ &= -2\Psi^K{}_{BDC}\epsilon^{K'}{}_{B'}(\chi_{KA}\epsilon_{K'A'} + \overline{\chi}_{K'A'}\epsilon_{KA})\epsilon^{A'B'} \\ &= -4\Psi^K{}_{BDC}\chi_{KA}. \end{aligned}$$

With analogous calculations for the remaining terms, the transvected equation (A.5) becomes

$$\begin{aligned} 4\Psi_{ABCD} &= -3\cdot 4(\Psi^K{}_{BCD}\chi_{KA} + \Psi^K{}_{ADC}\chi_{KB} + \Psi^K{}_{DBA}\chi_{KC} + \Psi^K{}_{CBA}\chi_{KD}) \\ &= -3\cdot 4\cdot 4\Psi_{(ABC}{}^K\chi_{D)K}. \end{aligned}$$

Adjusting for the different normalization, this matches our result $\Psi_{ABCD} = \Psi_{(ABC}{}^E\chi_{D)K}$ up to a sign.